

# MINIMAL REGULARITY SOLUTIONS OF SEMILINEAR GENERALIZED TRICOMI EQUATIONS

ZHUOPING RUAN, INGO WITT, AND HUICHENG YIN

**ABSTRACT.** We prove the local existence and uniqueness of minimal regularity solutions  $u$  of the semilinear generalized Tricomi equation  $\partial_t^2 u - t^m \Delta u = F(u)$  with initial data  $(u(0, \cdot), \partial_t u(0, \cdot)) \in \dot{H}^\gamma(\mathbb{R}^n) \times \dot{H}^{\gamma - \frac{2}{m+2}}(\mathbb{R}^n)$  under the assumption that  $|F(u)| \lesssim |u|^\kappa$  and  $|F'(u)| \lesssim |u|^{\kappa-1}$  for some  $\kappa > 1$ . Our results improve previous results of M. Beals [2] and of ourselves [15–17]. We establish Strichartz-type estimates for the linear generalized Tricomi operator  $\partial_t^2 - t^m \Delta$  from which the semilinear results are derived.

## 1. INTRODUCTION

In this paper, we are concerned with the local well-posedness problem for minimal regularity solutions  $u$  of the semilinear generalized Tricomi equation

$$\begin{cases} \partial_t^2 u - t^m \Delta u = F(u) & \text{in } (0, T) \times \mathbb{R}^n, \\ u(0, \cdot) = \varphi \in \dot{H}^\gamma(\mathbb{R}^n), \\ \partial_t u(0, \cdot) = \psi \in \dot{H}^{\gamma - \frac{2}{m+2}}(\mathbb{R}^n), \end{cases} \quad (1.1)$$

where  $n \geq 2$ ,  $m \in \mathbb{N}$ ,  $\gamma \in \mathbb{R}$ ,  $\Delta = \sum_{i=1}^n \partial_i^2$ , and  $T > 0$ . The nonlinearity  $F \in C^1(\mathbb{R})$  obeys the estimates

$$|F(u)| \lesssim |u|^\kappa, \quad |F'(u)| \lesssim |u|^{\kappa-1} \quad (1.2)$$

for some  $\kappa > 1$ . For  $n \geq 3$  and  $\kappa > \kappa_3$  (see below) we further assume that  $\kappa \in \mathbb{N}$  and  $F(u) = \pm u^\kappa$ .

Our main objective of this paper is to find the minimal number  $\gamma$  for which Eq. (1.1) under assumption (1.2) possesses a unique local solution  $u \in C([0, T], \dot{H}^\gamma(\mathbb{R}^n)) \cap L^s((0, T); L^q(\mathbb{R}^n))$  for certain  $s, q$  with  $\min\{s, q\} \geq \kappa$ . Then  $F(u) \in L^{s/\kappa}((0, T); L^{q/\kappa}(\mathbb{R}^n)) \subseteq L_{\text{loc}}^1((0, T) \times \mathbb{R}^n)$ , and Eq. (1.1) holds in distributions.

We first introduce notation used throughout this paper. Set

$$\mu_* = \frac{(m+2)n+2}{2}, \quad \kappa_* = \frac{\mu_*+2}{\mu_*-2} = \frac{(m+2)n+6}{(m+2)n-2},$$

$$\kappa_0 = 1 + \frac{6\mu_*+m}{\mu_*(m+2)n} \quad \text{if } n \geq 3 \text{ or } n = 2, m \geq 3,$$

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$$\kappa_1 = \begin{cases} 2 & \text{if } n = 2, m = 1, \\ \frac{(\mu_* + 2)(m + 2)(n - 1) + 8}{(\mu_* - 2)(m + 2)(n - 1) + 8} & \text{if } n \geq 3 \text{ or } n = 2, m \geq 2, \end{cases}$$

$$\kappa_2 = \frac{\mu_*(\mu_* + 2)(n - 1) - 2(n + 1)}{\mu_*(\mu_* - 2)(n - 1) - 2(n + 1)},$$

and

$$\kappa_3 = \frac{\mu_* - m}{\mu_* - m - 4} \quad \text{if } n \geq 3.$$

Note that  $\mu_*$  is the homogeneous dimension of the degenerate differential operator  $\partial_t^2 - t^m \Delta$  and  $\kappa_*$  is the power  $\kappa$  for which the equation  $\partial_t^2 u - t^m \Delta u = \pm |u|^{\kappa-1} u$  is conformally invariant. Note further that  $1 < \kappa_0 < \kappa_1 < \kappa_* < \kappa_2 < \kappa_3$  whenever it applies.

Now we state the main results of this paper.

**Theorem 1.1.** *Let  $n \geq 2$  and  $F$  be as above. Suppose further  $\kappa > \kappa_1$  and  $(\varphi, \psi) \in \dot{H}^\gamma(\mathbb{R}^n) \times \dot{H}^{\gamma - \frac{2}{m+2}}(\mathbb{R}^n)$ , where*

$$\gamma = \gamma(\kappa, m, n) = \begin{cases} \frac{n+1}{4} - \frac{n+1}{\mu_*(\kappa-1)} - \frac{m}{2\mu_*(m+2)} & \text{if } \kappa_1 < \kappa \leq \kappa_*, \\ \frac{n}{2} - \frac{1}{(m+2)(\kappa-1)} & \text{if } \kappa \geq \kappa_*. \end{cases} \quad (1.3)$$

Then problem (1.1) possesses a unique solution

$$u \in C([0, T]; \dot{H}^\gamma(\mathbb{R}^n)) \cap L^s((0, T); L^q(\mathbb{R}^n))$$

for some  $T > 0$ , where

$$\|u\|_{C([0, T]; \dot{H}^\gamma(\mathbb{R}^n))} + \|u\|_{L^s((0, T); L^q(\mathbb{R}^n))} \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma - \frac{2}{m+2}}(\mathbb{R}^n)} \quad (1.4)$$

and  $q = \mu_*(\kappa - 1)/2$ ,

$$\frac{1}{s} = \begin{cases} \frac{(m+2)(n-1)}{4} \left( \frac{1}{2} - \frac{1}{q} \right) + \frac{m}{4\mu_*} & \text{if } \kappa_1 < \kappa \leq \kappa_*, \\ 1/q & \text{if } \kappa \geq \kappa_*. \end{cases}$$

**Remark 1.2.** As a byproduct of the proof of Theorem 1.1, we see that problem (1.1) admits a unique global solution  $u \in C([0, \infty); \dot{H}^\gamma(\mathbb{R}^n)) \cap L^\infty((0, \infty); \dot{H}^\gamma(\mathbb{R}^n)) \cap L^{\frac{\mu_*(\kappa-1)}{2}}(\mathbb{R}_+ \times \mathbb{R}^n)$  in case  $n \geq 2$ ,  $\kappa \geq \kappa_*$  if  $(\varphi, \psi) = \varepsilon(u_0, u_1)$ ,  $(u_0, u_1) \in \dot{H}^\gamma(\mathbb{R}^n) \times \dot{H}^{\gamma - \frac{2}{m+2}}(\mathbb{R}^n)$ , and  $\varepsilon > 0$  is small (cf. 5.1.2 and 5.1.3 in the proof of Theorem 1.1 below). With a different argument, the global result  $u \in L^{\frac{\mu_*(\kappa-1)}{2}}(\mathbb{R}_+ \times \mathbb{R}^n)$  for problem (1.1) was obtained in [7].

**Remark 1.3.** For  $\gamma < \frac{n}{2} - \frac{4}{(m+2)(\kappa-1)}$ , one obtains ill-posedness for problem (1.1) by scaling. More specifically, if  $u = u(t, x)$  solves the Cauchy problem (1.1), where  $F(u) = \pm |u|^{\kappa-1} u$ , then

$$u_\varepsilon(t, x) = \varepsilon^{-\frac{2}{\kappa-1}} u(\varepsilon^{-1} t, \varepsilon^{-\frac{m+2}{2}} x), \quad \varepsilon > 0,$$

also solves (1.1), with  $u_\varepsilon(0, x) = \varphi_\varepsilon(x)$ ,  $\partial_t u_\varepsilon(0, x) = \psi_\varepsilon(x)$  for some resulting  $\varphi_\varepsilon, \psi_\varepsilon$ . Observe that

$$\frac{\|\varphi_\varepsilon\|_{\dot{H}^\gamma(\mathbb{R}^n)}}{\|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)}} = \frac{\|\psi_\varepsilon\|_{\dot{H}^\gamma(\mathbb{R}^n)}}{\|\psi\|_{\dot{H}^\gamma(\mathbb{R}^n)}} = \varepsilon^{\frac{m+2}{2}(\frac{n}{2}-\gamma) - \frac{2}{\kappa-1}},$$

and  $\frac{m+2}{2}(\frac{n}{2}-\gamma) - \frac{2}{\kappa-1} > 0$  for  $\gamma < \frac{n}{2} - \frac{4}{(m+2)(\kappa-1)}$ . Hence,  $\gamma < \frac{n}{2} - \frac{4}{(m+2)(\kappa-1)}$  implies that both the norm of the data  $(\varphi_\varepsilon, \psi_\varepsilon)$  and the lifespan  $T_\varepsilon = \varepsilon T$  of the solution  $u_\varepsilon$  go to zero as  $\varepsilon \rightarrow 0$ , where  $T$  is the lifespan of the solution  $u$ .

In case  $\kappa_* \leq \kappa < \kappa_2$ , as a supplement to Theorem 1.1, we consider the local existence and uniqueness of solutions  $u$  of problem (1.1) in the space  $C([0, T]; \dot{H}^\gamma(\mathbb{R}^n)) \cap L^s((0, T); L^q(\mathbb{R}^n))$  for certain  $s \neq q$ .

**Theorem 1.4.** *Let  $n \geq 2$ ,  $F$  be above,  $\gamma = \gamma(\kappa, m, n)$  be as in Theorem 1.1, and suppose that  $\kappa_* \leq \kappa < \kappa_2$ . Then the unique solution  $u$  of problem (1.1) also belongs to the space  $L^s((0, T); L^q(\mathbb{R}^n))$ , where*

$$\frac{1}{q} = \frac{1}{(m+2)(n-1)} \left( \frac{8}{\kappa-1} - \frac{m}{\mu_*} \right) - \frac{n-1}{2(n+1)}$$

and

$$\frac{1}{s} = \frac{(m+2)(n-1)}{4} \left( \frac{1}{2} - \frac{1}{q} \right) + \frac{m}{4\mu_*}.$$

Moreover, estimate (1.4) is satisfied.

If  $n \geq 3$  or  $n = 2, m \geq 3$ , then we find a number  $\gamma(\kappa, m, n)$  also for certain  $\kappa$  in the range  $1 < \kappa < \kappa_1$ .

**Theorem 1.5.** *Let  $n \geq 3$  or  $n = 2, m \geq 3$ . Let  $F$  be as above and  $\kappa_0 \leq \kappa < \kappa_1$ . In addition, let the exponent  $\gamma = \gamma(\kappa, m, n)$  in (1.1) be given by*

$$\gamma(\kappa, m, n) = \frac{n+1}{4} - \frac{n+1}{4\mu_*(m+2)} \cdot \frac{\mu_*(m+2)(n-1) + 12\mu_* + 2m}{2n\kappa - (n+1)} - \frac{m}{2\mu_*(m+2)}. \quad (1.5)$$

Then problem (1.1) possesses a unique solution  $u \in C([0, T]; \dot{H}^\gamma(\mathbb{R}^n)) \cap L^s((0, T); L^q(\mathbb{R}^n))$  for some  $T > 0$ , where

$$\frac{1}{q} = \frac{1}{2n\kappa - (n+1)} \left( \frac{n-1}{2} + \frac{6}{m+2} + \frac{m}{\mu_*(m+2)} \right)$$

and

$$\frac{1}{s} = \frac{(m+2)(n-1)}{4} \left( \frac{1}{2} - \frac{1}{q} \right) + \frac{m}{4\mu_*}.$$

Moreover, estimate (1.4) is satisfied.

**Remark 1.6.** Other than for the wave equation when  $m = 0$  (see also Remark 1.8 below), here  $\gamma$  can be negative in certain situations. In fact,  $\gamma(\kappa, m, n) < 0$  holds in the following cases:

- (i)  $\kappa_1 < \kappa < \frac{35}{17}$  ( $< \kappa_*$ ) if  $n = 2, m = 1$  and  $\kappa_1 < \kappa < \frac{13}{7}$  ( $< \kappa_*$ ) if  $n = 2, m = 2$  (see Theorem 1.1),

(ii)  $\kappa_0 < \kappa < \frac{\mu_*(\mu_*+2)(n+1)}{\mu_*(\mu_*-1)(n+1)-mn}$  ( $\leq \kappa_1$ ) if  $n \geq 3$  or  $n = 2, m \geq 3$  (see Theorem 1.5).

*Remark 1.7.* For initial data  $(\varphi, \psi)$  belonging to  $H^\gamma(\mathbb{R}^n) \times H^{\gamma-\frac{2}{m+2}}(\mathbb{R}^n)$ , where  $\gamma \geq \gamma(\kappa, m, n)$ , Theorems 1.1, 1.4, and 1.5 remain valid.

*Remark 1.8.* For  $m = 0$ , (1.1) becomes

$$\begin{cases} \partial_t^2 u - \Delta u = F(u) & \text{in } (0, T) \times \mathbb{R}^n, \\ u(0, \cdot) = \varphi \in \dot{H}^\gamma(\mathbb{R}^n), \\ \partial_t u(0, \cdot) = \psi \in \dot{H}^{\gamma-1}(\mathbb{R}^n), \end{cases}$$

while the exponents  $\kappa_*$ ,  $\kappa_0$ ,  $\kappa_1$ ,  $\kappa_2$ , and  $\kappa_3$  are

$$\begin{aligned} \kappa_* &= \frac{n+3}{n-1}, \quad \kappa_2 = \frac{(n+1)^2 - 6}{(n-1)^2 - 2}, \\ \kappa_1 &= \frac{(n+1)^2}{(n-1)^2 + 4} \quad \text{if } n \geq 3, \\ \kappa_0 &= 1 + \frac{3}{n}, \quad \kappa_3 = \frac{n+1}{n-3} \quad \text{if } n \geq 4. \end{aligned}$$

For  $n \geq 3$ ,  $\gamma$  defined in (1.3) equals

$$\gamma(\kappa, 0, n) = \begin{cases} \frac{n+1}{4} - \frac{1}{\kappa-1} & \text{if } \kappa_1 < \kappa \leq \kappa_*, \\ \frac{n}{2} - \frac{2}{\kappa-1} & \text{if } \kappa \geq \kappa_*, \end{cases} \quad (1.6)$$

whereas, for  $n \geq 4$ ,  $\gamma$  defined in (1.5) equals

$$\gamma(\kappa, 0, n) = \frac{n+1}{4} - \frac{(n+1)(n+5)}{4} \frac{1}{2n\kappa - (n+1)}. \quad (1.7)$$

Note that the numbers in (1.6) and (1.7) are exactly those in (2.1) and (2.5) of [10]. In that paper, [10], the local existence problem for minimal regularity solutions of the semilinear wave equation was systematically studied. The results were achieved by establishing Strichartz-type estimates for the linear wave operator  $\partial_t^2 - \Delta$ . Under certain restrictions on the nonlinearity  $F(u, \nabla u)$ , for the more general semilinear wave equation

$$\begin{cases} \partial_t^2 u - \Delta u = F(u, \nabla u), \\ u(0, x) = \varphi(x), \quad \partial_t u(0, x) = \psi(x), \end{cases}$$

many remarkable results on the ill-posedness or well-posedness problem on the local existence of low regularity solutions have been obtained, see [8–10, 14, 18, 21] and the reference therein.

*Remark 1.9.* There are some essential differences between degenerate hyperbolic equations and strictly hyperbolic equations. Amongst others, the symmetry group is smaller (see [11]) and there is a loss of regularity for the linear Cauchy problem (see e.g. [4, 22]). Therefore, as compared to the semilinear wave equation, a more delicate analysis is required when one studies minimal regularity results for the semilinear generalized Tricomi equation in the degenerate hyperbolic region.

The Tricomi equation (i.e., Eq. (1.1) for  $n = 1, m = 1$ ) were first studied by Tricomi [23] who initiated work on boundary value problems for linear partial differential operators of mixed elliptic-hyperbolic type. So far, these equations have been extensively studied in bounded domains under suitable boundary conditions and several applications to transonic flow problems were given (see [3, 6, 13, 23] and the references therein). Conservation laws for equations of mixed type were derived by Lupo and Payne [11, 12]. In [17], we established the local solvability for low regularity solutions of the semilinear equation  $\partial_t^2 u - t^m \Delta u = F(u)$ , where  $n \geq 2, m \in \mathbb{N}$  is odd, in the domain  $(-T, T) \times \mathbb{R}^n$  for some  $T > 0$ . In [1, 24, 26], fundamental solutions for the linear Tricomi operator and the linear generalized Tricomi operator have been explicitly computed. In case  $n = 2$  and  $m = 1$ , Beals [2] obtained the local existence of the solution  $u$  of the equation  $\partial_t^2 u - t \Delta u = F(u)$  with initial data of  $H^s$ -regularity, where  $s > n/2$ . For the equation  $\partial_t^2 u - t^m \Delta u = a(t)F(u)$ , where  $n \geq 2, m \in \mathbb{N}$  is even, and both  $a$  and  $F$  are of power type, Yadjjian [25] obtained global existence and uniqueness for small data solutions provided the solution  $v$  of the linear problem  $\partial_t^2 v - t^m \Delta v = 0$  fulfills  $t^\beta v \in C([0, \infty); L^q(\mathbb{R}^n))$  for certain  $\beta, q$  depending on  $n, m$ , and the powers occurring in  $a$  and  $F$ . In [15, 16], for the semilinear generalized Tricomi equation  $\partial_t^2 u - t^m \Delta u = F(u)$  with initial data of a special structure, i.e., homogeneous of degree 0 or piecewise smooth along a hyperplane, we obtained local existence and uniqueness via establishing  $L^\infty$  estimates on the solutions  $v$  of the linear equation  $\partial_t^2 v - t^m \Delta v = g$ . Note that when the nonlinear term  $F(u)$  is of power type, for higher and higher powers of  $\kappa$ , these  $L^\infty$  estimates are basically required to guarantee existence. In this paper, where the initial data in  $\dot{H}^\gamma(\mathbb{R}^n)$  is of no special structure and  $\gamma$  is minimal to guarantee local well-posedness of problem (1.1), the arguments of [15, 16] fail. Inspired by the methods in [10], however, we are able to overcome the technical difficulties related to degeneracy and low regularity and eventually obtain the local well-posedness of problem (1.1).

We first study the linear problem

$$\begin{cases} \partial_t^2 u - t^m \Delta u = f(t, x) & \text{in } (0, T) \times \mathbb{R}^n, \\ u(0, \cdot) = \varphi(x), \quad \partial_t u(0, \cdot) = \psi(x) \end{cases} \quad (1.8)$$

and establish Strichartz-type estimates of the form

$$\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L_t^s L_x^q(S_T)} \leq C \left( \|\varphi\|_{\dot{H}^\gamma} + \|\psi\|_{\dot{H}^{\gamma - \frac{2}{m+2}}} + \|f\|_{L_t^r L_x^p(S_T)} \right) \quad (1.9)$$

for certain  $s, q, r, p$  (for details see below) and some constant  $C = C(T, \gamma, s, q, r, p) > 0$ , where  $S_T = (0, T) \times \mathbb{R}^n$ . Note that, by scaling, a necessary condition for this estimate in case  $T = \infty$  to hold is

$$\frac{(m+2)n}{2} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{1}{r} - \frac{1}{s} = 2. \quad (1.10)$$

In doing so, in Section 2, we introduce certain Fourier integral operators  $W (= W^0)$  and  $W^\alpha$  for  $\alpha \in \mathbb{C}$ . These operators depend on a parameter  $\mu \geq 2$ , introduced in (2.15), which plays an auxiliary role for the linear problems and agrees with the homogeneous dimension  $\mu_*$  when applied to the semilinear problems. Along with the operators  $W$  and  $W^\alpha$  we also consider their parts  $W_j$  and  $W_j^\alpha$ , respectively, resulting from a dyadic decomposition of frequency space. Continuity of the operators  $W_j$  and  $W_j^\alpha$  between function spaces which holds uniformly in  $j$  ultimately provides linear estimates on the solutions  $u$  of Eq. (1.8).

In Section 3, we prove boundedness of the operators  $W_j^\alpha$  from  $L_t^r L_x^p(\mathbb{R}_+^{1+n})$  to  $L_t^{r'} L_x^{p'}(\mathbb{R}_+^{1+n})$  (see Theorem 3.1) and from  $L_t^r L_x^p(\mathbb{R}_+^{1+n})$  to  $L_t^\infty L_x^2(\mathbb{R}_+^{1+n})$  (see Theorem 3.4), where  $\mu$  has to satisfy the lower bound  $\mu \geq \max\{2, m/2\}$ . Combining Theorem 3.1 and Stein's analytic interpolation theorem, we show boundedness of the operators  $W_j^\alpha$  from  $L^q(\mathbb{R}_+^{1+n})$  to  $L^{p_0}(\mathbb{R}_+^{1+n})$ , where  $q_0 \leq q \leq \infty$  (see Theorem 3.6). Through an additional dyadic decomposition now with respect to the time variable  $t$ , using Theorems 3.1 and 3.6 together with interpolation, we prove boundedness of the operators  $W_j$  from  $L_t^r L_x^p((0, T) \times \mathbb{R}^n)$  to  $L_t^s L_x^q((0, T) \times \mathbb{R}^n)$  for any  $T > 0$  (see Theorems 3.7 and 3.8), where  $\mu$  has to satisfy the new lower bounds  $\mu \geq \mu_*$  (Theorem 3.7) and  $\mu \geq \max\{2, mn/2\}$  (Theorem 3.8), respectively.

In the sequel, we shall use the following notation:

$$\frac{1}{p_0} = \frac{1}{2} + \frac{2\mu - m}{\mu(2\mu_* - m)}, \quad \frac{1}{p_1} = \frac{1}{2} + \frac{2\mu - m}{\mu(m+2)(n-1)}, \quad \frac{1}{p_2} = \frac{2}{p_0} - \frac{1}{p_1}.$$

Note that

$$1 < p_1 \leq p_0 \leq p_2 \leq 2 \quad \text{if } n \geq 3 \text{ or } n = 2, m \geq 2,$$

while  $1 \leq p_1$  in case of  $n = 2$  and  $m = 1$  requires  $\mu = 2$  (and then  $p_1 = 1$ ). For  $1 \leq p \leq 2$ ,  $p'$  denotes the conjugate exponent of  $p$  defined by  $\frac{1}{p} + \frac{1}{p'} = 1$ . Further,  $q_\ell$  denotes  $p'_\ell$  for  $\ell = 0, 1, 2$ , while  $q_0^*$  equals  $q_0$  when  $\mu = \mu_*$  (see Remark 4.2). We often abbreviate function spaces  $C_t^0 \dot{H}_x^\gamma(S_T) = C([0, T]; \dot{H}^\gamma(\mathbb{R}^n))$ ,  $L_t^r L_x^p(S_T) = L^r((0, T); L^p(\mathbb{R}^n))$ , and  $A \lesssim B$  means that  $A \leq CB$  holds for some generic constant  $C > 0$ .

The paper is organized as follows: In Section 2, we define a class of Fourier integral operators associated with the linear generalized Tricomi operator  $\partial_t^2 - t^m \Delta$  in  $\mathbb{R}_+ \times \mathbb{R}^n$ . Then, in Section 3, we establish a series of mixed-norm space-time estimates for those Fourier integral operators. These estimates are applied, in Section 4, to obtain Strichartz-type estimates for the solutions of the linear generalized Tricomi equation which in turn, in Section 5, allow us to prove the local existence and uniqueness results for problem (1.1).

## 2. SOME PRELIMINARIES

In this section, we first recall an explicit formula for the solution of the linear generalized Tricomi equation obtained in [22] and then apply it to define a class of Fourier integral operators which will play a key role in proving our main results.

Consider the linear generalized Tricomi equation

$$\begin{cases} \partial_t^2 u - t^m \Delta u = f(t, x) & \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \\ u(0, \cdot) = \varphi, \quad \partial_t u(0, \cdot) = \psi. \end{cases} \quad (2.1)$$

Its solution  $u$  can be written as  $u = v + w$ , where  $v$  solves the homogeneous equation

$$\begin{cases} \partial_t^2 v - t^m \Delta v = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \\ v(0, \cdot) = \varphi, \quad \partial_t v(0, \cdot) = \psi \end{cases} \quad (2.2)$$

and  $w$  solves the inhomogeneous equation with zero initial data

$$\begin{cases} \partial_t^2 w - t^m \Delta w = f(t, x) & \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \\ w(0, \cdot) = \partial_t w(0, \cdot) = 0. \end{cases} \quad (2.3)$$

Recall that (see [22] or [25]) the solutions  $v$  and  $w$  of problems (2.2) and (2.3) can be expressed as

$$v(t, x) = V_0(t, D_x)\varphi(x) + V_1(t, D_x)\psi(x)$$

and

$$w(t, x) = \int_0^t (V_1(t, D_x)V_0(\tau, D_x) - V_0(t, D_x)V_1(\tau, D_x)) f(\tau, x) d\tau, \quad (2.4)$$

where the symbols  $V_j(t, \xi)$  ( $j = 0, 1$ ) of the Fourier integral operators  $V_j(t, D_x)$  are

$$\begin{cases} V_0(t, \xi) = e^{-z/2} \Phi\left(\frac{m}{2(m+2)}, \frac{m}{m+2}; z\right), \\ V_1(t, \xi) = te^{-z/2} \Phi\left(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}; z\right) \end{cases} \quad (2.5)$$

with  $z = 2i\phi(t)|\xi|$  and  $\phi(t) = (2/(m+2))t^{(m+2)/2}$ . Here,  $\Phi(a, c; z)$  is the confluent hypergeometric function which is an analytic function of  $z$ . Recall (see [5, page 254]) that

$$\frac{d^n}{dz^n} \Phi(a, c; z) = \frac{(a)_n}{(c)_n} \Phi(a+n, c+n; z), \quad (2.6)$$

where  $(a)_0 = 1$ ,  $(a)_n = a(a+1) \dots (a+n-1)$ . In addition (see [25, (3.5)-(3.7)]), for  $0 < \arg(z) < \pi$ , one has that

$$e^{-z/2} \Phi(a, c; z) = \frac{\Gamma(c)}{\Gamma(a)} e^{z/2} H_+(a, c; z) + \frac{\Gamma(c)}{\Gamma(c-a)} e^{-z/2} H_-(a, c; z), \quad (2.7)$$

where

$$\begin{aligned} H_+(a, c; z) &= \frac{e^{-i\pi(c-a)}}{e^{i\pi(c-a)} - e^{-i\pi(c-a)}} \frac{1}{\Gamma(c-a)} z^{a-c} \int_{\infty}^{(0+)} e^{-\theta} \theta^{c-a-1} \left(1 - \frac{\theta}{z}\right)^{a-1} d\theta, \\ H_-(a, c; z) &= \frac{1}{e^{i\pi a} - e^{-i\pi a}} \frac{1}{\Gamma(a)} z^{-a} \int_{\infty}^{(0+)} e^{-\theta} \theta^{a-1} \left(1 + \frac{\theta}{z}\right)^{c-a-1} d\theta. \end{aligned}$$

Moreover, it holds that

$$\begin{aligned} \left| \partial_{\xi}^{\beta} (H_+(a, c; 2i\phi(t)|\xi|)) \right| &\lesssim (\phi(t)|\xi|)^{a-c} (1 + |\xi|)^{-|\beta|} \quad \text{if } \phi(t)|\xi| \geq 1, \\ \left| \partial_{\xi}^{\beta} (H_-(a, c; 2i\phi(t)|\xi|)) \right| &\lesssim (\phi(t)|\xi|)^{-a} (1 + |\xi|)^{-|\beta|} \quad \text{if } \phi(t)|\xi| \geq 1. \end{aligned} \quad (2.8)$$

Choose  $\eta \in C_c^{\infty}(\mathbb{R}_+)$  such that  $0 \leq \eta \leq 1$  with  $\eta(r) = 1$  if  $r \leq 1$  and  $\eta(r) = 0$  if  $r \geq 2$ . Then from (2.5) and (2.7), we can write

$$V_0(t, D_x)\varphi(x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi - \phi(t)|\xi|)} b_1(t, \xi) \hat{\varphi}(\xi) d\xi + \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)|\xi|)} b_2(t, \xi) \hat{\varphi}(\xi) d\xi \quad (2.9)$$

and

$$V_1(t, D_x)\psi(x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi - \phi(t)|\xi|)} b_3(t, \xi) \hat{\psi}(\xi) d\xi + \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)|\xi|)} b_4(t, \xi) \hat{\psi}(\xi) d\xi, \quad (2.10)$$

where

$$\begin{aligned} b_1(t, \xi) &= \eta(\phi(t)|\xi|) \Phi\left(\frac{m}{2(m+2)}, \frac{m}{m+2}; z\right) + (1 - \eta(\phi(t)|\xi|)) H_- \left(\frac{m}{2(m+2)}, \frac{m}{m+2}; z\right), \\ b_2(t, \xi) &= (1 - \eta(\phi(t)|\xi|)) H_+ \left(\frac{m}{2(m+2)}, \frac{m}{m+2}; z\right), \end{aligned}$$

and

$$\begin{aligned} b_3(t, \xi) &= t\eta(\phi(t)|\xi|) \Phi\left(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}; z\right) + t(1 - \eta(\phi(t)|\xi|)) H_- \left(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}; z\right), \\ b_4(t, \xi) &= t(1 - \eta(\phi(t)|\xi|)) H_+ \left(\frac{m+4}{2(m+2)}, \frac{m+4}{m+2}; z\right). \end{aligned}$$

We can also write

$$\begin{aligned} \int_0^t V_0(t, D_x) V_1(\tau, D_x) f(\tau, x) d\tau &= \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) + \phi(\tau))|\xi|)} b_2(t, \xi) b_4(\tau, \xi) \hat{f}(\tau, \xi) d\xi d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau))|\xi|)} b_2(t, \xi) b_3(\tau, \xi) \hat{f}(\tau, \xi) d\xi d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi - (\phi(t) + \phi(\tau))|\xi|)} b_1(t, \xi) b_3(\tau, \xi) \hat{f}(\tau, \xi) d\xi d\tau \quad (2.11) \\ &\quad + \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi - (\phi(t) - \phi(\tau))|\xi|)} b_1(t, \xi) b_4(\tau, \xi) \hat{f}(\tau, \xi) d\xi d\tau \end{aligned}$$

and

$$\begin{aligned} \int_0^t V_1(t, D_x) V_0(\tau, D_x) f(\tau, x) d\tau &= \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) + \phi(\tau))|\xi|)} b_4(t, \xi) b_2(\tau, \xi) \hat{f}(\tau, \xi) d\xi d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi - (\phi(t) - \phi(\tau))|\xi|)} b_3(t, \xi) b_2(\tau, \xi) \hat{f}(\tau, \xi) d\xi d\tau \\ &\quad + \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi - (\phi(t) + \phi(\tau))|\xi|)} b_3(t, \xi) b_1(\tau, \xi) \hat{f}(\tau, \xi) d\xi d\tau \quad (2.12) \\ &\quad + \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau))|\xi|)} b_4(t, \xi) b_1(\tau, \xi) \hat{f}(\tau, \xi) d\xi d\tau, \end{aligned}$$

where  $\hat{f}(\tau, \xi)$  is the Fourier transform of  $f(\tau, x)$  with respect to the variable  $x$  and  $d\xi = (2\pi)^{-n} d\xi$ .

In view of the analyticity of  $\Phi(a, c; z)$  with respect to the variable  $z$ , identity (2.6), and estimates (2.8), we have that, for  $(t, \xi) \in \mathbb{R}_+^{1+n}$ ,

$$|\partial_\xi^\beta b_\ell(t, \xi)| \lesssim (1 + \phi(t)|\xi|)^{-\frac{m}{2(m+2)}} |\xi|^{-|\beta|}, \quad \ell = 1, 2, \quad (2.13)$$



and

$$|\partial_\xi^\beta b_\ell(t, \xi)| \lesssim t(1 + \phi(t)|\xi|)^{-\frac{m+4}{2(m+2)}} |\xi|^{-|\beta|}, \quad \ell = 3, 4. \quad (2.14)$$

Thus, for  $\ell = 1, 2, k = 3, 4, \mu \geq 2, t, \tau > 0$ , and  $\xi \in \mathbb{R}^n$ , one has from (2.13) and (2.14) that

$$\begin{aligned} |\partial_\xi^\beta (b_k(t, \xi) b_\ell(\tau, \xi))| &\lesssim t(1 + \phi(t)|\xi|)^{-\frac{m+4}{2(m+2)}} (1 + \phi(\tau)|\xi|)^{-\frac{m}{2(m+2)}} |\xi|^{-|\beta|} \\ &\lesssim (1 + \phi(t)|\xi|)^{-\frac{m}{2(m+2)}} (1 + \phi(\tau)|\xi|)^{-\frac{m}{2(m+2)}} |\xi|^{-\frac{2}{m+2}-|\beta|} \\ &\lesssim (1 + |\phi(t) - \phi(\tau)| |\xi|)^{-\frac{m}{\mu(m+2)}} |\xi|^{-\frac{2}{m+2}-|\beta|}. \end{aligned} \quad (2.15)$$

Furthermore, estimates (2.13)-(2.15) yield that, for  $\ell = 1, 2, k = 3, 4$  or  $\ell = 3, 4, k = 1, 2$  and for  $\mu \geq 2, t, s > 0$ , and  $\xi \in \mathbb{R}^n$ , one has

$$\begin{aligned} \left| \partial_\xi^\beta \left( \int_t^\infty \overline{b_\ell(\tau, \xi) b_k(t, \xi)} \partial_\tau (b_\ell(\tau, \xi) b_k(s, \xi)) d\tau \right) \right| \\ \lesssim (1 + |\phi(t) - \phi(s)| |\xi|)^{-\frac{m}{\mu(m+2)}} |\xi|^{-\frac{4}{m+2}-|\beta|} \end{aligned} \quad (2.16)$$

and

$$\begin{aligned} \left| \partial_\xi^\beta \left( \int_s^\infty \overline{b_\ell(\tau, \xi) b_k(t, \xi)} \partial_\tau (b_\ell(\tau, \xi) b_k(s, \xi)) d\tau \right) \right| \\ \lesssim (1 + |\phi(t) - \phi(s)| |\xi|)^{-\frac{m}{\mu(m+2)}} |\xi|^{-\frac{4}{m+2}-|\beta|}. \end{aligned} \quad (2.17)$$

In order to study the function  $w$  in (2.4), in view of (2.11), (2.12) and (2.15)-(2.17), it suffices to consider, for a given  $\mu \geq 2$ , the Fourier integral operator  $W$ ,

$$Wf(t, x) = \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(s))|\xi|)} b(t, s, \xi) \hat{f}(s, \xi) d\xi ds, \quad (2.18)$$

where  $b \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n)$  satisfies

(i) for  $t, s > 0$  and  $\xi \in \mathbb{R}^n$ ,

$$|\partial_\xi^\beta b(t, s, \xi)| \lesssim (1 + |\phi(t) - \phi(s)| |\xi|)^{-\frac{m}{\mu(m+2)}} |\xi|^{-\frac{2}{m+2}-|\beta|}, \quad (2.19)$$

(ii) for  $t, s > 0$  and  $\xi \in \mathbb{R}^n$ ,

$$\left| \partial_\xi^\beta \left( \int_t^\infty \overline{b(\tau, t, \xi)} \partial_\tau b(\tau, s, \xi) d\tau \right) \right| \lesssim (1 + |\phi(t) - \phi(s)| |\xi|)^{-\frac{m}{\mu(m+2)}} |\xi|^{-\frac{4}{m+2}-|\beta|} \quad (2.20)$$

and

$$\left| \partial_\xi^\beta \left( \int_s^\infty \overline{b(\tau, t, \xi)} \partial_\tau b(\tau, s, \xi) d\tau \right) \right| \lesssim (1 + |\phi(t) - \phi(s)| |\xi|)^{-\frac{m}{\mu(m+2)}} |\xi|^{-\frac{4}{m+2}-|\beta|}. \quad (2.21)$$

Let  $\Theta \in C_c^\infty(\mathbb{R}_+)$  satisfy  $\text{supp } \Theta \subseteq [1/2, 2]$  and

$$\sum_{j=-\infty}^{\infty} \Theta(t/2^j) = 1 \quad \text{for } t > 0.$$

Then, as in [10], for  $j \in \mathbb{Z}$  and  $\alpha \in \mathbb{C}$ , we define dyadic operators  $W_j$  and  $W_j^\alpha$ ,

$$W_j f(t, x) = \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(s))|\xi|)} b_j(t, s, \xi) \hat{f}(s, \xi) d\xi ds$$

and

$$W_j^\alpha f(t, x) = \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(s))|\xi|)} b_j(t, s, \xi) \hat{f}(s, \xi) \frac{d\xi}{|\xi|^\alpha} ds, \quad (2.22)$$

where  $b_j(t, s, \xi) = \Theta(|\xi|/2^j) b(t, s, \xi)$ ; here  $b \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n)$  satisfies the estimates (2.19)-(2.21).

Littlewood-Paley theory gives us a relationship between  $Wf$  and  $W_j f (= W_j^0 f)$ , which will play an important role in our arguments in Section 4.

**Proposition 2.1.** *Let  $n \geq 2$ . For  $1 < p \leq 2$ ,  $1 \leq r \leq 2$ ,  $2 \leq q < \infty$ , and  $2 \leq s \leq \infty$ , let*

$$\|W_j f\|_{L_t^s L_x^q} \lesssim \|f\|_{L_t^r L_x^p} \quad (2.23)$$

*hold uniformly in  $j$ . Then*

$$\|Wf\|_{L_t^s L_x^q} \lesssim \|f\|_{L_t^r L_x^p}.$$

*Proof.* This is actually an application of Lemma 3.8 of [10]. For the sake of completeness, we give the proof here. By Littlewood-Paley theory (see, e. g., [20]), for any  $1 < \rho < \infty$ ,

$$\|Wf(t, \cdot)\|_{L^\rho(\mathbb{R}^n)} \lesssim \left\| \left( \sum_{j=-\infty}^{\infty} |W_j f(t, \cdot)|^2 \right)^{1/2} \right\|_{L^\rho(\mathbb{R}^n)} \lesssim \|Wf(t, \cdot)\|_{L^\rho(\mathbb{R}^n)}.$$

Together with the Minkowski inequality, this yields

$$\|Wf\|_{L_t^s L_x^q} \lesssim \left( \sum_{j=-\infty}^{\infty} \|W_j f\|_{L_t^s L_x^q}^2 \right)^{1/2} \quad (2.24)$$

and

$$\left( \sum_{j=-\infty}^{\infty} \|W_j f\|_{L_t^r L_x^p}^2 \right)^{1/2} \lesssim \|Wf\|_{L_t^r L_x^p}. \quad (2.25)$$

Notice that

$$f = \sum_{k=-\infty}^{\infty} f_k,$$

where  $f_k(\tau, x) = \Theta(\tau/2^k) f(\tau, x)$ . Therefore, for some  $M_0 \in \mathbb{N}$ ,

$$\begin{aligned} \|Wf\|_{L_t^s L_x^q}^2 &\lesssim \sum_{j=-\infty}^{\infty} \|W_j f\|_{L_t^s L_x^q}^2 && \text{(by (2.24))} \\ &= \sum_{j=-\infty}^{\infty} \left\| W_j \left( \sum_{|j-k| \leq M_0} f_k \right) \right\|_{L_t^s L_x^q}^2 && \text{(due to the compact support of } \Theta) \\ &\lesssim \sum_{j=-\infty}^{\infty} \left( \sum_{|j-k| \leq M_0} \|W_j f_k\|_{L_t^s L_x^q} \right)^2 && \text{(by Minkowski inequality)} \\ &\lesssim \sum_{j=-\infty}^{\infty} \sum_{|j-k| \leq M_0} \|f_k\|_{L_t^r L_x^p}^2 && \text{(by (2.23))} \end{aligned}$$

$$\lesssim \sum_{j=-\infty}^{\infty} \|f_j\|_{L_t^r L_x^p}^2 \lesssim \|f\|_{L_t^r L_x^p}^2. \quad (\text{by (2.25)}),$$

which completes the proof of Proposition 2.1.  $\square$

### 3. MIXED-NORM ESTIMATES FOR A CLASS OF FOURIER INTEGRAL OPERATORS

In this section, for  $j \in \mathbb{Z}$ ,  $\alpha \in \mathbb{C}$ , and  $\mu \geq 2$ , we shall study mixed norm estimates for the class of Fourier integral operators  $W_j^\alpha$  defined in (2.22).

We start by considering the boundedness of the operator  $W_j^\alpha$  from  $L_t^r L_x^p$  to  $L_t^{r'} L_x^{p'}$ , where  $1 < r, p \leq 2$ . We denote  $\lambda_j = 2^j$ . *All the following estimates hold uniformly in  $j$ .*

**Theorem 3.1.** *Let  $n \geq 2$  and  $\mu \geq \max\{2, m/2\}$ . Then:*

(i) *For  $\max\{p_1, 1\} < p \leq 2$  and*

$$\frac{1}{r} = 1 - \frac{m}{4\mu} - \frac{(m+2)(n-1)}{4} \left( \frac{1}{p} - \frac{1}{2} \right), \quad (3.1)$$

*we have that*

$$\|W_j^\alpha f\|_{L_t^{r'} L_x^{p'}(\mathbb{R}_+^{1+n})} \lesssim \lambda_j^{\left(\frac{1}{p}-\frac{1}{2}\right)(n+1) - \frac{m}{\mu(m+2)} - \frac{2}{m+2} - \operatorname{Re} \alpha} \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})}. \quad (3.2)$$

*Consequently,*

$$\|W_j^\alpha f\|_{L_t^{r'} L_x^{p'}(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})} \quad \text{if } \operatorname{Re} \alpha = \left( \frac{1}{p} - \frac{1}{2} \right) (n+1) - \frac{m}{\mu(m+2)} - \frac{2}{m+2}. \quad (3.3)$$

(ii) *For  $p_1 > 1$  and  $1 < p < p_1$ , we have that*

$$\|W_j^\alpha f\|_{L_t^2 L_x^{p'}(\mathbb{R}_+^{1+n})} \lesssim \lambda_j^{n\left(\frac{2}{p}-1\right) - \frac{4}{m+2} - \operatorname{Re} \alpha} \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})}. \quad (3.4)$$

*In particular,*

$$\|W_j^\alpha f\|_{L_t^2 L_x^{p'}(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})} \quad \text{if } \operatorname{Re} \alpha = n \left( \frac{2}{p} - 1 \right) - \frac{4}{m+2}. \quad (3.5)$$

To prove Theorem 3.1, for fixed  $t, \tau > 0$ , we first consider the operator  $B_j^\alpha$ ,

$$B_j^\alpha f(t, \tau, x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau))|\xi|)} b_j(t, \tau, \xi) \hat{f}(\tau, \xi) \frac{d\xi}{|\xi|^\alpha}.$$

**Lemma 3.2.** *Let  $n \geq 2$  and  $1 \leq p \leq 2$ . Then, for  $t, \tau > 0$ ,*

$$\begin{aligned} \|B_j^\alpha f(t, \tau, \cdot)\|_{L^{p'}(\mathbb{R}^n)} &\lesssim \lambda_j^{\left(\frac{1}{p}-\frac{1}{2}\right)(n+1) - \frac{m}{\mu(m+2)} - \frac{2}{m+2} - \operatorname{Re} \alpha} \\ &\quad \times \left( \lambda_j^{-\frac{2}{m+2}} + |t - \tau| \right)^{-(m+2)\left(\frac{1}{p}-\frac{1}{2}\right) - \frac{n-1}{2} - \frac{m}{2\mu}} \|f(\tau, \cdot)\|_{L^p(\mathbb{R}^n)}. \end{aligned} \quad (3.6)$$

*Proof.* Denote

$$K_j^\alpha(t, \tau, x, y) = \int_{\mathbb{R}^n} e^{i((x-y) \cdot \xi + (\phi(t) - \phi(\tau))|\xi|)} b_j(t, \tau, \xi) \frac{d\xi}{|\xi|^\alpha}. \quad (3.7)$$

Then  $B_j^\alpha f$  can be written as

$$B_j^\alpha f(t, \tau, x) = \int_{\mathbb{R}^n} K_j^\alpha(t, \tau, x, y) f(\tau, y) dy.$$

Since  $\text{supp}_\xi b_j \subseteq \{\xi \in \mathbb{R}^n \mid \lambda_j/2 \leq |\xi| \leq 2\lambda_j\}$ , we have from (2.19) that

$$|\partial_\xi^\beta b_j(t, \tau, \xi)| \lesssim \lambda_j^{-\frac{m}{\mu(m+2)} - \frac{2}{m+2} - |\beta|} \left( \lambda_j^{-\frac{2}{m+2}} + |t - \tau| \right)^{-\frac{m}{2\mu}}. \quad (3.8)$$

We now apply (3.8) to derive estimate (3.6) by Plancherel's theorem when  $p = 2$  and by the stationary phase method when  $p = 1$ . By interpolation, we then obtain (3.6) for  $1 < p < 2$ .

Indeed, it follows from Plancherel's theorem that

$$\begin{aligned} \|B_j^\alpha f(t, \tau, \cdot)\|_{L_x^2(\mathbb{R}^n)} &= \|e^{i(\phi(t) - \phi(\tau))|\xi|} b_j(t, \tau, \xi) \hat{f}(\tau, \xi) |\xi|^{-\alpha}\|_{L_\xi^2(\mathbb{R}^n)} \\ &\lesssim \lambda_j^{-\frac{m}{\mu(m+2)} - \frac{2}{m+2} - \text{Re } \alpha} \left( \lambda_j^{-\frac{2}{m+2}} + |t - \tau| \right)^{-\frac{m}{2\mu}} \|f(\tau, \cdot)\|_{L^2(\mathbb{R}^n)}. \end{aligned} \quad (3.9)$$

On the other hand, by the stationary phase method (see e.g. [19, Lemma 7.2.4]), we have that, for any  $N \geq 0$ ,

$$\begin{aligned} |K_j^\alpha(t, \tau, x, y)| &\lesssim \lambda_j^n (1 + |\phi(t) - \phi(\tau)| \lambda_j)^{-\frac{n-1}{2}} \left( \lambda_j^{-\frac{2}{m+2}} + |t - \tau| \right)^{-\frac{m}{2\mu}} \\ &\quad \times \lambda_j^{-\frac{m}{\mu(m+2)} - \frac{2}{m+2} - \text{Re } \alpha} \left( 1 + \lambda_j ||x - y| - |\phi(t) - \phi(\tau)|| \right)^{-N} \\ &\lesssim \lambda_j^{\frac{n+1}{2} - \frac{m}{\mu(m+2)} - \frac{2}{m+2} - \text{Re } \alpha} \left( \lambda_j^{-\frac{2}{m+2}} + |t - \tau| \right)^{-\frac{(m+2)(n-1)}{4} - \frac{m}{2\mu}} \\ &\quad \times \left( 1 + \lambda_j ||x - y| - |\phi(t) - \phi(\tau)|| \right)^{-N}. \end{aligned} \quad (3.10)$$

Choosing  $N = 0$  in (3.10) gives

$$\begin{aligned} \|(B_j^\alpha f)(t, \tau, \cdot)\|_{L^\infty(\mathbb{R}^n)} &\leq \|K_j^\alpha(t, \tau, \cdot, \cdot)\|_{L_{x,y}^\infty} \|f(\tau, \cdot)\|_{L^1(\mathbb{R}^n)} \\ &\lesssim \lambda_j^{\frac{n+1}{2} - \frac{m}{\mu(m+2)} - \frac{2}{m+2} - \text{Re } \alpha} \left( \lambda_j^{-\frac{2}{m+2}} + |t - \tau| \right)^{-\frac{(m+2)(n-1)}{4} - \frac{m}{2\mu}} \|f(\tau, \cdot)\|_{L^1(\mathbb{R}^n)}. \end{aligned} \quad (3.11)$$

Interpolation between (3.9) and (3.11) yields (3.6) in case  $1 \leq p \leq 2$  which completes the proof of estimate (3.6).  $\square$

**Proof of Theorem 3.1.** Now we return to the proof of Theorem 3.1. From (3.7), we have

$$W_j^\alpha f(t, x) = \int_0^t (B_j^\alpha f)(t, \tau, x) d\tau. \quad (3.12)$$

Using Minkowski's inequality and estimate (3.6), we thus have that

$$\begin{aligned} \|W_j^\alpha f(t, \cdot)\|_{L^{p'}(\mathbb{R}^n)} &\lesssim \lambda_j^{\left(\frac{1}{p} - \frac{1}{2}\right)(n+1) - \frac{m}{\mu(m+2)} - \frac{2}{m+2} - \text{Re } \alpha} \\ &\quad \times \int_0^\infty \left( \lambda_j^{-\frac{2}{m+2}} + |t - \tau| \right)^{-(m+2)\left(\frac{1}{p} - \frac{1}{2}\right)\frac{n-1}{2} - \frac{m}{2\mu}} \|f(\tau, \cdot)\|_{L^p(\mathbb{R}^n)} d\tau. \end{aligned} \quad (3.13)$$

1) **Case  $\max\{p_1, 1\} < p \leq 2$ .** In this case, we have  $1 < r < 2$ . Note that

$$\frac{1}{r} - \frac{1}{r'} = -(m+2) \left( \frac{1}{p} - \frac{1}{2} \right) \frac{n-1}{2} - \frac{m}{2\mu} + 1.$$

Then it follows from the Hardy-Littlewood-Sobolev theorem and (3.13) that estimate (3.2) holds.

2) **Case  $p_1 > 1$  and  $1 < p < p_1$ .** In this case,

$$(m+2) \left( \frac{1}{p} - \frac{1}{2} \right) \frac{n-1}{2} + \frac{m}{2\mu} > 1.$$

Thus,

$$\sup_{t>0} \int_0^\infty \left( \lambda_j^{-\frac{2}{m+2}} + |t-\tau| \right)^{-(m+2)\left(\frac{1}{p}-\frac{1}{2}\right)\frac{n-1}{2}-\frac{m}{2\mu}} d\tau < \infty,$$

which together with Schur's lemma and (3.13) yields (3.4).  $\square$

We would like to stress that in the proof of Theorem 3.1 only condition (2.19) on the function  $b \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n)$  was used, whereas the conditions (2.20) and (2.21) were not required,

*Remark 3.3.* Note that the adjoint operator  $(W_j^\alpha)^*$  of  $W_j^\alpha$  is of the form

$$(W_j^\alpha)^* f(t, x) = \int_t^\infty \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau))|\xi|)} \overline{b_j(\tau, t, \xi)} \hat{f}(\tau, \xi) \frac{d\xi}{|\xi|^\alpha} d\tau. \quad (3.14)$$

By duality, we infer from Theorem 3.1 that

$$\|(W_j^\alpha)^* f\|_{L_t^{r'} L_x^{p'}(\mathbb{R}_+^{1+n})} \lesssim \lambda_j^{\left(\frac{1}{p}-\frac{1}{2}\right)(n+1)-\frac{m}{\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re} \alpha} \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})} \quad (3.15)$$

if  $\max\{p_1, 1\} < p \leq 2$  and

$$\|(W_j^\alpha f)^*\|_{L_t^2 L_x^{p'}(\mathbb{R}_+^{1+n})} \lesssim \lambda_j^{n\left(\frac{2}{p}-1\right)-\frac{4}{m+2}-\operatorname{Re} \alpha} \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})}. \quad (3.16)$$

if  $p_1 > 1$  and  $1 < p < p_1$ . Here,  $r$  is given in (3.1).

As an application of Theorem 3.1, we obtain the boundedness of the operator  $W_j^\alpha$  from  $L_t^r L_x^p$  to  $L_t^\infty L_x^2$ , where  $1 < r, p \leq 2$ .

**Theorem 3.4.** *Let  $n \geq 2$  and  $\mu \geq \max\{2, m/2\}$ . Then:*

(i) *For  $\max\{p_1, 1\} < p \leq 2$  and  $r$  be as in (3.1), we have that*

$$\|W_j^\alpha f\|_{L_t^\infty L_x^2(\mathbb{R}_+^{1+n})} \lesssim \lambda_j^{\left(\frac{1}{p}-\frac{1}{2}\right)\frac{n+1}{2}-\frac{m}{2\mu(m+2)}-\frac{2}{m+2}-\operatorname{Re} \alpha} \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})}. \quad (3.17)$$

*Consequently,*

$$\|W_j^\alpha f\|_{L_t^\infty L_x^2(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})} \quad \text{if } \operatorname{Re} \alpha = \left( \frac{1}{p} - \frac{1}{2} \right) \frac{n+1}{2} - \frac{m}{2\mu(m+2)} - \frac{2}{m+2}. \quad (3.18)$$

(ii) *For  $p_1 > 1$  and  $1 < p < p_1$ , we have that*

$$\|W_j^\alpha f\|_{L_t^\infty L_x^2(\mathbb{R}_+^{1+n})} \lesssim \lambda_j^{n\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{3}{m+2}-\operatorname{Re} \alpha} \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})}. \quad (3.19)$$

In particular,

$$\|W_j^\alpha f\|_{L_t^\infty L_x^2(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})} \quad \text{if } \operatorname{Re} \alpha = n \left( \frac{1}{p} - \frac{1}{2} \right) - \frac{3}{m+2}. \quad (3.20)$$

*Proof.* For given  $j \in \mathbb{Z}$  and  $\alpha \in \mathbb{C}$ , denote  $U = W_j^\alpha f$ . Then from (2.22) we have

$$U(t) = \int_0^t e^{i(\phi(t)-\phi(\tau))\sqrt{-\Delta}} b_j(t, \tau, D_x) (-\Delta)^{-\alpha/2} f(\tau) d\tau,$$

where  $b_j(t, \tau, D_x)$  is the pseudodifferential operator with full symbol  $b_j(t, \tau, \xi)$ . Then  $U(t)$  solves the Cauchy problem

$$\begin{cases} i\partial_t U(t) = -t^{m/2} \sqrt{-\Delta} U(t) + i b_j(t, t, D_x) (-\Delta)^{-\alpha/2} f(t) \\ \quad + i \int_0^t e^{i(\phi(t)-\phi(\tau))\sqrt{-\Delta}} \partial_t b_j(t, \tau, D_x) (-\Delta)^{-\alpha/2} f(\tau) d\tau, \\ U(0) = 0. \end{cases}$$

Multiplying by  $\overline{U(t)}$  and then integrating over  $\mathbb{R}^n$  yields

$$\begin{aligned} i \langle \partial_t U(t), U(t) \rangle &= -t^{m/2} \langle \sqrt{-\Delta} U(t), U(t) \rangle + i \langle b_j(t, t, D_x) (-\Delta)^{-\alpha/2} f(t), U(t) \rangle \\ &\quad + i \left\langle \int_0^t e^{i(\phi(t)-\phi(\tau))\sqrt{-\Delta}} \partial_t b_j(t, \tau, D_x) (-\Delta)^{-\alpha/2} f(\tau) d\tau, U(t) \right\rangle, \end{aligned}$$

and, therefore,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|U(t)\|^2 &= \operatorname{Re} \left\langle \int_0^t e^{i(\phi(t)-\phi(\tau))\sqrt{-\Delta}} \partial_t b_j(t, \tau, D_x) (-\Delta)^{-\alpha/2} f(\tau) d\tau, U(t) \right\rangle \\ &\quad + \operatorname{Re} \langle b_j^*(t, t, D_x) (-\Delta)^{-\alpha/2} U(t), f(t) \rangle. \end{aligned}$$

Consequently,

$$\begin{aligned} \|U(s)\|^2 &= 2 \operatorname{Re} \int_0^s \left\langle \int_0^t e^{i(\phi(t)-\phi(\tau))\sqrt{-\Delta}} \partial_t b_j(t, \tau, D_x) (-\Delta)^{-\alpha/2} f(\tau) d\tau, U(t) \right\rangle dt \\ &\quad + 2 \operatorname{Re} \int_0^s \langle b_j^*(t, t, D_x) (-\Delta)^{-\alpha/2} U(t), f(t) \rangle dt \\ &\lesssim \left| \int_0^s \int_{\mathbb{R}^n} L_j^\alpha f(t, x) \overline{W_j^\alpha f(t, x)} dx dt \right| \\ &\quad + \left| \int_0^s \int_{\mathbb{R}^n} b_j^*(t, t, D_x) W_j^{2\alpha} f(t, x) \overline{f(t, x)} dx dt \right| \\ &= \text{I} + \text{II}, \end{aligned}$$

where

$$\begin{aligned} \text{I} &= \left| \int_0^s \int_{\mathbb{R}^n} L_j^\alpha f(t, x) \overline{W_j^\alpha f(t, x)} dx dt \right| \\ \text{II} &= \left| \int_0^s \int_{\mathbb{R}^n} b_j^*(t, t, D_x) W_j^{2\alpha} f(t, x) \overline{f(t, x)} dx dt \right|, \end{aligned}$$

and

$$L_j^\alpha f(t, x) = \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau))|\xi|)} \partial_t b_j(t, \tau, \xi) \hat{f}(\tau, \xi) \frac{d\xi}{|\xi|^\alpha} d\tau.$$

From (2.19), one has that, for any fixed  $t > 0$ ,  $b_j(t, t, D_x) \in \Psi^{-\frac{2}{m+2}}(\mathbb{R}^n)$ , and then  $b_j^*(t, t, D_x) \in \Psi^{-\frac{2}{m+2}}(\mathbb{R}^n)$ , which yields that the term II is essentially

$$\left| \int_0^s \int_{\mathbb{R}^n} (W_j^{2\alpha + \frac{2}{m+2}} f)(t, x) \overline{f(t, x)} dx dt \right|,$$

and thus by application of Theorem 3.1 it follows that

$$\text{II} \lesssim \begin{cases} \lambda_j^{(n+1)(\frac{1}{p}-\frac{1}{2}) - \frac{m}{\mu(m+2)} - \frac{4}{m+2} - 2\text{Re } \alpha} \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})}^2 & \text{if } \max\{p_1, 1\} < p \leq 2, \\ \lambda_j^{n(\frac{2}{p}-1) - \frac{6}{m+2} - 2\text{Re } \alpha} \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})}^2 & \text{if } p_1 > 1 \text{ and } 1 < p < p_1. \end{cases} \quad (3.21)$$

As for the term I, note that

$$\text{I} = \left| \int_0^s \int_{\mathbb{R}^n} (W_j^\alpha)^* L_j^\alpha f(t, x) \overline{f(t, x)} dx dt \right| \leq \|(W_j^\alpha)^* L_j^\alpha f\|_{L_t^{\rho'} L_x^{p'}(\mathbb{R}_+^{1+n})} \|f\|_{L_t^\rho L_x^p(\mathbb{R}_+^{1+n})}.$$

For any  $t > 0$ , we have from (3.14) that

$$\begin{aligned} & (W_j^\alpha)^* L_j^\alpha f(t, x) \\ &= \int_t^\infty \int_0^\tau \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(s))|\xi|)} \overline{b_j(\tau, t, \xi)} \partial_\tau b_j(\tau, s, \xi) \hat{f}(s, \xi) \frac{d\xi}{|\xi|^{2\alpha}} ds d\tau \\ &= \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(s))|\xi|)} \left( \int_t^\infty \overline{b_j(\tau, t, \xi)} \partial_\tau b_j(\tau, s, \xi) d\tau \right) \hat{f}(s, \xi) \frac{d\xi}{|\xi|^{2\alpha}} ds \\ &\quad + \int_t^\infty \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(s))|\xi|)} \left( \int_s^\infty \overline{b_j(\tau, t, \xi)} \partial_\tau b_j(\tau, s, \xi) d\tau \right) \hat{f}(s, \xi) \frac{d\xi}{|\xi|^{2\alpha}} ds. \end{aligned} \quad (3.22)$$

Due to conditions (2.19)-(2.21), one has that the first and second term in (3.22) are essentially  $W_j^{2\alpha + \frac{2}{m+2}} f$  and  $(W_j^{2\alpha + \frac{2}{m+2}})^* f$ , respectively, where  $b \in C^\infty(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n)$  satisfies condition (2.19). Then, by applying Theorem 3.1 and estimates (3.15) and (3.16), we have that

$$\text{I} \lesssim \begin{cases} \lambda_j^{(n+1)(\frac{1}{p}-\frac{1}{2}) - \frac{m}{\mu(m+2)} - \frac{4}{m+2} - 2\text{Re } \alpha} \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})}^2 & \text{if } \max\{p_1, 1\} < p \leq 2, \\ \lambda_j^{n(\frac{2}{p}-1) - \frac{6}{m+2} - 2\text{Re } \alpha} \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})}^2 & \text{if } p_1 > 1 \text{ and } 1 < p < p_1, \end{cases}$$

which together with (3.21) yields that

$$\|U(t)\|^2 \lesssim \begin{cases} \lambda_j^{(n+1)(\frac{1}{p}-\frac{1}{2}) - \frac{m}{\mu(m+2)} - \frac{4}{m+2} - 2\text{Re } \alpha} \|f\|_{L_t^r L_x^p(\mathbb{R}_+^{1+n})}^2 & \text{if } \max\{p_1, 1\} < p \leq 2, \\ \lambda_j^{n(\frac{2}{p}-1) - \frac{6}{m+2} - 2\text{Re } \alpha} \|f\|_{L_t^2 L_x^p(\mathbb{R}_+^{1+n})}^2 & \text{if } p_1 > 1 \text{ and } 1 < p < p_1. \end{cases}$$

Note that  $\|W_j^\alpha f(t, \cdot)\|_{L^2(\mathbb{R}^n)} = \|U(t)\|$ . Therefore, we have obtained estimates (3.17)-(3.20) which completes the proof of Theorem 3.4.  $\square$

*Remark 3.5.* With similar arguments as in the proof of Theorem 3.4, we have from Theorem 3.1 and estimates (3.15) and (3.16) that the operator  $(W_j^\alpha)^*$  also satisfies the estimates (3.17)-(3.20).

Note that if  $r = p$  for  $r$  defined in (3.1), then  $r = p = p_0$ . Combining Theorem 3.1 and the kernel estimate (3.10), we obtain boundedness of the operator  $W_j^\alpha$  from  $L^{p_0}(\mathbb{R}_+^{1+n})$  to  $L^q(\mathbb{R}_+^{1+n})$  for certain  $\alpha \in \mathbb{C}$  when  $q_0 \leq q \leq \infty$ .

**Theorem 3.6.** *Let  $\mu \geq \max\{2, m/2\}$  and  $q_0 \leq q \leq \infty$ . Then*

$$\|W_j^\alpha f\|_{L^q(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L^{p_0}(\mathbb{R}_+^{1+n})}, \quad (3.23)$$

where

$$\operatorname{Re} \alpha = n - \frac{2}{m+2} - \left(n + \frac{2}{m+2}\right) \left(\frac{1}{q} + \frac{1}{q_0}\right).$$

*Proof.* (i) *Case  $q = q_0$ .* Note that

$$n - \frac{2}{q_0} \left(n + \frac{2}{m+2}\right) = \left(\frac{1}{p_0} - \frac{1}{2}\right) (n+1) - \frac{m}{\mu(m+2)}.$$

An application of (3.3) with  $r = p$  yields that

$$\|W_j^\alpha f\|_{L^{q_0}(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L^{p_0}(\mathbb{R}_+^{1+n})}, \quad \operatorname{Re} \alpha = n - \frac{2}{m+2} - \frac{2}{q_0} \left(n + \frac{2}{m+2}\right). \quad (3.24)$$

(ii) *Case  $q = \infty$ .* In order to derive (3.23), it suffices to show that the integral kernel  $K_j^\alpha$  defined in (3.7) satisfies

$$\sup_{(t,x) \in \mathbb{R}_+^{1+n}} \int_{\mathbb{R}_+^{1+n}} |K_j^\alpha(t, \tau, x, y)|^{q_0} d\tau dy < \infty, \quad \operatorname{Re} \alpha = n - \frac{2}{m+2} - \frac{1}{q_0} \left(n + \frac{2}{m+2}\right). \quad (3.25)$$

In fact, from (3.7) we have

$$W_j^\alpha f(t, x) = \int_0^t \int_{\mathbb{R}^n} K_j^\alpha(t, \tau, x, y) f(\tau, y) dy d\tau.$$

By Hölder's inequality, then

$$\|W_j^\alpha f\|_{L^\infty(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L^{p_0}(\mathbb{R}_+^{1+n})}, \quad \operatorname{Re} \alpha = n - \frac{2}{m+2} - \frac{1}{q_0} \left(n + \frac{2}{m+2}\right). \quad (3.26)$$

Now it remains to derive estimate (3.25). In fact, due to the kernel estimate (3.10), for any  $N > n$  and  $\alpha \in \mathbb{C}$  with  $\operatorname{Re} \alpha = n - \frac{2}{m+2} - \frac{1}{q_0} \left(n + \frac{2}{m+2}\right)$ , we have by (3.10)

$$\int_{\mathbb{R}_+^{1+n}} |K_j^\alpha(t, \tau, x, y)|^{q_0} d\tau dy$$



$$\begin{aligned}
&\lesssim \lambda_j^{\left(\frac{n+1}{2} - \operatorname{Re} \alpha - \frac{m}{\mu(m+2)} - \frac{2}{m+2}\right)q_0} \int_0^\infty \left(\lambda_j^{-\frac{2}{m+2}} + |t - \tau|\right)^{-\left(\frac{(m+2)(n-1)}{4} + \frac{m}{2\mu}\right)q_0} d\tau \\
&\quad \times \int_{\mathbb{R}^n} \left(1 + \lambda_j ||x - y| - |\phi(t) - \phi(\tau)||\right)^{-N} dy \\
&\lesssim \lambda_j^{\left(\frac{n+1}{2} - \operatorname{Re} \alpha - \frac{m}{\mu(m+2)} - \frac{2}{m+2}\right)q_0} \int_0^\infty \left(\lambda_j^{-\frac{2}{m+2}} + |t - \tau|\right)^{-\left(\frac{(m+2)(n-1)}{4} + \frac{m}{2\mu}\right)q_0} d\tau \\
&\quad \times \lambda_j^{-1} \int_0^\infty (1+r)^{-N} \left(\lambda_j^{-1} r + |\phi(t) - \phi(\tau)|\right)^{n-1} dr \\
&= \lambda_j^{\left(\frac{n+1}{2} - \operatorname{Re} \alpha - \frac{m}{\mu(m+2)} - \frac{2}{m+2}\right)q_0 - 1} \\
&\quad \times \int_0^\infty \left(\lambda_j^{-\frac{2}{m+2}} + |t - \tau|\right)^{-\left(\frac{(m+2)(n-1)}{4} + \frac{m}{2\mu}\right)q_0} \left(\lambda_j^{-1} + |\phi(t) - \phi(\tau)|\right)^{n-1} d\tau \\
&\quad \times \int_0^\infty (1+r)^{-N} \left(\frac{r + \lambda_j |\phi(t) - \phi(\tau)|}{1 + \lambda_j |\phi(t) - \phi(\tau)|}\right)^{n-1} dr \\
&\lesssim \lambda_j^{\left(\frac{n+1}{2} - \operatorname{Re} \alpha - \frac{m}{\mu(m+2)} - \frac{2}{m+2}\right)q_0 - 1} \int_0^\infty \left(\lambda_j^{-\frac{2}{m+2}} + |t - \tau|\right)^{-\left(\frac{(m+2)(n-1)}{4} + \frac{m}{2\mu}\right)q_0 + \frac{(m+2)(n-1)}{2}} d\tau \\
&\lesssim \lambda_j^{\left(n - \operatorname{Re} \alpha - \frac{2}{m+2}\right)q_0 - n - \frac{2}{m+2}} = 1,
\end{aligned}$$

and hence (3.25) holds.

(iii) *Case*  $q_0 < q < \infty$ . Applying Stein's interpolation theorem, one obtains that estimate (3.23) holds by interpolating between estimates (3.24) and (3.26).  $\square$

Now we consider boundedness of the operator  $W_j$  from  $L_t^r L_x^p(S_T)$  to  $L_t^s L_x^q(S_T)$ , where  $1/p$  is symmetric around  $1/p_0$ .

**Theorem 3.7.** *Let  $n \geq 2$ . Further let  $p_1 < p < p_2$  if  $n = 2$ ,  $m \geq 2$  or if  $n \geq 3$ , and  $1 < p < \frac{7\mu}{4\mu-2}$  if  $n = 2$ ,  $m = 1$ . Then, for any  $\mu \geq \mu_*$  and  $T > 0$ ,*

$$\|W_j f\|_{L_t^s L_x^q(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)}, \quad (3.27)$$

where  $r$  is defined as in (3.1) and

$$\begin{cases} \frac{1}{q} = \frac{1}{p} - \frac{4}{(m+2)(n+1)} \left(1 + \frac{m}{2\mu}\right), \\ \frac{1}{s} = \frac{(m+2)(n-1)}{4} \left(\frac{1}{2} - \frac{1}{q}\right) + \frac{m}{4\mu}. \end{cases} \quad (3.28)$$

*Proof.* Since  $1/p$  is symmetric around  $1/p_0$ , by duality it suffices to consider the case  $\max\{p_1, 1\} < p \leq p_0$ .

In order to derive (3.27), we now need a further dyadic decomposition with respect to the time variable  $t$ . Choose a function  $\eta \in C_c^\infty(\mathbb{R}_+)$  such that  $0 \leq \eta \leq 1$ ,  $\operatorname{supp} \eta \subseteq [1/2, 2]$ , and

$$\sum_{\ell=-\infty}^{\infty} \eta(2^{-\ell} t) = 1.$$

Let us fix  $\lambda = 2^j$  and set

$$\eta_0(t) = \sum_{k \leq 0} \eta(\lambda 2^{-k}t), \quad \eta_\ell(t) = \eta(\lambda 2^{-\ell}t) \quad \text{for } \ell \in \mathbb{N}.$$

Then,

$$W_j f(t, x) = \sum_{k=0}^{\infty} G_k f(t, x),$$

where

$$G_k f(t, x) = \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau))|\xi|)} \eta_k(t - \tau) b_j(t, \tau, \xi) \hat{f}(\tau, \xi) d\xi d\tau. \quad (3.29)$$

Hence, to derive (3.27), it suffices to show that, for any  $k \in \mathbb{N}_0$ ,

$$\|G_k f\|_{L_t^s L_x^q(S_T)} \lesssim 2^{-\varepsilon_p k} \|f\|_{L_t^r L_x^p(S_T)} \quad (3.30)$$

for some  $\varepsilon_p > 0$ . From (3.1) and (3.28), we know that

$$\frac{(m+2)n}{2} \left( \frac{1}{p} - \frac{1}{q} \right) + \frac{1}{r} - \frac{1}{s} = 2.$$

Due to scaling invariance, we need to consider only the case  $\lambda = 1$  (by a change of variable if  $\lambda \neq 1$ ). Repeating the arguments which are used to prove (3.2), we get that, for any  $k \in \mathbb{N}_0$ ,

$$\|G_k f\|_{L_t^{r'} L_x^{p'}(S_T)} \lesssim 2^{-k((m+2)(\frac{1}{p}-\frac{1}{2})\frac{n-1}{2}+\frac{m}{2\mu})} \|f\|_{L_t^r L_x^p(S_T)}. \quad (3.31)$$

Note that  $(m+2)(\frac{1}{p}-\frac{1}{2})\frac{n-1}{2}+\frac{m}{2\mu} > \frac{1}{3}$ , since  $p \leq p_0$ .

Furthermore, an immediate consequence of (3.17) for  $\alpha = 0$  is

$$\|G_k f\|_{L_t^\infty L_x^2(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)},$$

and thus, for any  $1 < \rho < \infty$ ,

$$\|G_k f\|_{L_t^\rho L_x^2(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)}. \quad (3.32)$$

Choose

$$\theta = \frac{4p(2\mu+m)}{\mu(m+2)(n+1)(2-p)} - 1. \quad (3.33)$$

Then  $0 \leq \theta \leq 1$  and, for the number  $q$  from (3.28),

$$\frac{1}{q} = \frac{\theta}{p'} + \frac{1-\theta}{2}.$$

For  $s$  from (3.28) and  $\theta$  from (3.33), we define  $s_0$  by

$$2 \left( \frac{1}{s} - \frac{1}{s_0} \right) = \theta \left( (m+2) \left( \frac{1}{p} - \frac{1}{2} \right) \frac{n-1}{2} + \frac{m}{2\mu} \right)$$

and then set  $\rho = \rho_*$  such that

$$\frac{1}{s_0} = \frac{\theta}{r'} + \frac{1-\theta}{\rho_*}.$$

Since  $2 < s < s_0$ , by interpolating between (3.31) and (3.32) when  $\rho = \rho_*$ , we obtain that

$$\|G_k f\|_{L_t^{s_0} L_x^q(S_T)} \lesssim 2^{-2k(\frac{1}{s}-\frac{1}{s_0})} \|f\|_{L_t^r L_x^p(S_T)}. \quad (3.34)$$

Let  $\{I_\ell\}$  be non-overlapping intervals of side length  $2^k$  and  $\bigcup_\ell I_\ell = \mathbb{R}_+$ , and denote by  $\chi_I$  the characteristic function of  $I$ . In view of (3.29) and the compact support of  $\eta_k$ , we have that if  $f(t, x) = 0$  for  $t \notin I_\ell$ , then  $G_k f(t, x) = 0$  for  $t \notin I_\ell^*$ , where  $I_\ell^*$  is the interval with the same center as  $I_\ell$  but of side length  $C_0 2^k$  with some constant  $C_0 = C_0(\eta) > 0$ . Thus, from Minkowski's inequality

$$\|G_k f(t, \cdot)\|_{L^q(\mathbb{R}^n)}^s \leq \left( \sum_\ell \|G_k(\chi_{I_\ell} f)(t, \cdot)\|_{L^q(\mathbb{R}^n)} \right)^s \lesssim \sum_\ell \|G_k(\chi_{I_\ell} f)(t, \cdot)\|_{L^q(\mathbb{R}^n)}^s, \quad (3.35)$$

Denote  $\overline{I_\ell^*} = I_\ell^* \cap (0, T)$ . Estimate (3.35) together with Hölder's inequality and (3.34) yields that, for any  $k \in \mathbb{N}_0$ ,

$$\begin{aligned} \|G_k f\|_{L_t^s L_x^q(S_T)}^s &\lesssim \sum_\ell \|G_k(\chi_{I_\ell} f)\|_{L_t^s L_x^q(\overline{I_\ell^*} \times \mathbb{R}^n)}^s \\ &\lesssim \sum_\ell |\overline{I_\ell^*}|^{1-s/s_0} \|G_k(\chi_{I_\ell} f)\|_{L_t^{s_0} L_x^q(\overline{I_\ell^*} \times \mathbb{R}^n)}^s \\ &\lesssim 2^{k(1-s/s_0)} 2^{-2ks(1/s-1/s_0)} \sum_\ell \|\chi_{I_\ell} f\|_{L_t^r L_x^p(S_T)}^s \\ &\lesssim 2^{-k(1-s/s_0)} \|f\|_{L_t^r L_x^p(S_T)}. \end{aligned}$$

Therefore, we get estimate (3.30) with  $\varepsilon_p = 1 - s/s_0$  and, hence, (3.27) holds.  $\square$

By a similar argument as in the proof of Theorem 3.7, we obtain the boundedness of operator  $W_j$  from  $L_t^2 L_x^p(S_T)$  to  $L_t^s L_x^q(S_T)$  when  $p_1 > 1$  and  $1 < p < p_1$ .

**Theorem 3.8.** *Let  $n \geq 3$  or  $n = 2$ ,  $m \geq 2$ . Suppose  $1 < p < p_1$ . Then, for  $\mu \geq \max\{2, mn/2\}$  and  $T > 0$ , we have that*

$$\|W_j f\|_{L_t^s L_x^q(S_T)} \lesssim \|f\|_{L_t^2 L_x^p(S_T)}, \quad (3.36)$$

where

$$\begin{cases} \frac{1}{q} = \frac{2n}{p(n+1)} - \frac{n-1}{2(n+1)} - \frac{m+6\mu}{\mu(m+2)(n+1)}, \\ \frac{1}{s} = (m+2) \left( \frac{1}{2} - \frac{1}{q} \right) \left( \frac{n-1}{4} \right) + \frac{m}{4\mu}. \end{cases} \quad (3.37)$$

*Proof.* Note that when  $1 < p < p_1$ , we have

$$(m+2) \left( \frac{1}{p} - \frac{1}{2} \right) \left( \frac{n-1}{2} \right) + \frac{m}{2\mu} > 1.$$

Then we can apply similar arguments as in the proof Theorem 3.7 to obtain (3.36). We omit the details.  $\square$

*Remark 3.9.* By similar arguments as above one can show that adjoints  $(W_j)^*$  of  $W_j$  also satisfy estimates (3.27) and (3.36), respectively, under assumptions (3.28) and (3.37).

#### 4. MIXED-NORM ESTIMATES FOR THE LINEAR GENERALIZED TRICOMI EQUATION

In this section, based on the mixed-norm space-time estimates of the Fourier integral operators  $W_j^\alpha$  obtained in Section 3, we shall establish Strichartz-type estimates for the linear generalized Tricomi equation.

First we consider the inhomogeneous equation, i.e., problem (2.3).

**Theorem 4.1.** *Let  $n \geq 2$ . Suppose  $w$  is a solution of (2.3) in  $S_T$  for some  $T > 0$ . Then:*

(i) *For  $\mu \geq \mu_*$ ,*

$$\|w\|_{L_t^s L_x^q(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)} \quad (4.1)$$

*provided that  $p_1 < p < p_2$  if  $n \geq 3$  or  $n = 2, m \geq 2$ , and  $1 < p < 7\mu/(4\mu - 2)$  if  $n = 2$  and  $m = 1$ . Here  $r = r(p, \mu)$  is as in (3.1) and  $q$  and  $s$  are taken from (3.28).*

(ii) *For  $\mu \geq \max\{2, m/2\}$ ,*

$$\|w\|_{L^q(S_T)} \lesssim \| |D_x|^{\gamma - \gamma_0} f \|_{L^{p_0}(S_T)}, \quad q_0 \leq q < \infty, \quad (4.2)$$

where

$$\begin{cases} \gamma = \gamma(m, n, q) = \frac{n}{2} - \frac{1}{q} \left( n + \frac{2}{m+2} \right), \\ \gamma_0 = \gamma_0(m, n, \mu) = \frac{1}{q_0} \left( n + \frac{2}{m+2} \right) + \frac{2}{m+2} - \frac{n}{2}. \end{cases} \quad (4.3)$$

(iii) *For  $\mu \geq \max\{2, m/2\}$ ,  $\max\{p_1, 1\} < p \leq 2$ , and  $0 \leq t \leq T$ ,*

$$\|w(t, \cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} \lesssim \|f\|_{L_t^r L_x^p(S_T)}, \quad (4.4)$$

where  $r = r(p, \mu)$  is defined in (3.1) and

$$\gamma = \gamma(m, n, \mu, p) = \frac{2}{m+2} + \frac{m}{2\mu(m+2)} - \left( \frac{1}{p} - \frac{1}{2} \right) \frac{n+1}{2}.$$

(iv) *For  $\mu \geq \max\{2, m/2\}$ ,  $\gamma \in \mathbb{R}$ , and  $0 \leq t \leq T$ ,*

$$\|w(t, \cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} \lesssim \| |D_x|^{\gamma - \gamma_0} f \|_{L^{p_0}(S_T)}, \quad (4.5)$$

where  $\gamma_0$  is from (4.3).

**Remark 4.2.** If we choose  $\mu = \mu_*$ , then

$$p_0 = p_0^* = \frac{2\mu_*}{\mu_* + 2}, \quad q_0 = q_0^* = \frac{2\mu_*}{\mu_* - 2},$$

and for  $\gamma$  and  $\gamma_0$  defined in (4.3),

$$\gamma(m, n, q_0^*) = \gamma_0(m, n, \mu_*) = \frac{1}{m+2}.$$

Thus, we have from (4.2) that

$$\|w\|_{L^{q_0^*}(S_T)} \lesssim \|f\|_{L^{p_0^*}(S_T)},$$

which, for any  $\rho \in \mathbb{R}$ , together with  $[|D_x|^\rho, \partial_t^2 - t^m \Delta] = 0$  implies that

$$\| |D_x|^\rho w \|_{L^{q_0^*}(S_T)} \lesssim \| |D_x|^\rho f \|_{L^{p_0^*}(S_T)}.$$

**Proof of Theorem 4.1.**

(i) One obtains (4.1) by applying Proposition 2.1 and Theorem 3.7 directly.

(ii) For  $\alpha \in \mathbb{C}$ , the Fourier transform of  $|D_x|^\alpha f(t, x)$  with respect to the variable  $x$  is  $|\xi|^\alpha \hat{f}(t, \xi)$ . Thus, we can write  $W_j f$  as

$$W_j f(t, x) = \int_0^t \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau))|\xi|)} \Theta(|\xi|/2^j) b(t, \tau, \xi) (\widehat{|D_x|^\alpha f})(\tau, \xi) |\xi|^{-\alpha} d\xi d\tau$$

and  $W_j(f) = W_j^\alpha(|D_x|^\alpha f)$ . Therefore, applying Theorem 3.6, we get that

$$\|W_j f\|_{L^q(S_T)} = \|W_j^{\gamma-\gamma_0}(|D_x|^{\gamma-\gamma_0} f)\|_{L^q(S_T)} \lesssim \| |D_x|^{\gamma-\gamma_0} f \|_{L^{p_0}(S_T)},$$

which together with Proposition 2.1 yields (4.2).

(iii) Note that  $[|D_x|^\gamma, \partial_t^2 - t^m \Delta] = 0$  and then

$$(\partial_t^2 - t^m \Delta)(|D_x|^\gamma w) = |D_x|^\gamma f. \quad (4.6)$$

From (ii) we know that  $W_j(|D_x|^\gamma f) = W_j^{-\gamma}(f)$ . Thus, for  $\gamma = \frac{2}{m+2} + \frac{m}{2\mu(m+2)} - \left(\frac{1}{p} - \frac{1}{2}\right) \frac{n+1}{2}$ , we have from estimate (3.18) that

$$\|W_j(|D_x|^\gamma f)(t, \cdot)\|_{L^2(\mathbb{R}^n)} = \|W_j^{-\gamma} f(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L_t^r L_x^p}.$$

Thus, by (4.6) and Proposition 2.1 it follows that

$$\|(|D_x|^\gamma w)(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L_t^r L_x^p},$$

which together with Plancherel's theorem implies that

$$\|w(t, \cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} = \| |\xi|^\gamma \hat{w}(t, \xi) \|_{L_\xi^2(\mathbb{R}^n)} = \|(|D_x|^\gamma w)(t, \cdot)\|_{L_x^2(\mathbb{R}^n)} \lesssim \|f\|_{L_t^r L_x^p},$$

and estimate (4.4) holds.

(iv) From (ii) we also know that

$$W_j(g) = W_j^{-\gamma_0}(|D_x|^{-\gamma_0} g).$$

In (3.1), we have  $r = p = p_0$  when  $r = p$ . Estimate (3.18) for

$$\alpha = -\gamma_0 = \left(\frac{1}{p_0} - \frac{1}{2}\right) \frac{n+1}{2} - \frac{m}{2\mu(m+2)} - \frac{2}{m+2}$$

with  $p = p_0$  yields that

$$\|W_j(g)(t, \cdot)\|_{L^2(\mathbb{R}^n)} = \|W_j^{-\gamma_0}(|D_x|^{-\gamma_0} g)(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \| |D_x|^{-\gamma_0} g \|_{L^{p_0}(S_T)},$$

and then, for  $g = |D_x|^\gamma f$ , where  $\gamma \in \mathbb{R}$ ,

$$\|W_j(|D_x|^\gamma f)(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \| |D_x|^{\gamma-\gamma_0} f \|_{L^{p_0}(S_T)}. \quad (4.7)$$

Therefore, one has from Plancherel's theorem, Proposition 2.1, (4.6), and (4.7) that

$$\|w(t, \cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} = \|(|D_x|^\gamma w)(t, \cdot)\|_{L^2(\mathbb{R}^n)} \lesssim \| |D_x|^{\gamma-\gamma_0} f \|_{L^{p_0}(S_T)}$$

Hence, estimate (4.5) holds.  $\square$

In case  $n \geq 2$  and  $m \geq 2$  if  $n = 2$ , we have a more complete set of inequalities for the solution of the linear generalized Tricomi equation.

**Theorem 4.3.** *Let  $n \geq 3$  or  $n = 2$ ,  $m \geq 2$ . Suppose  $w$  solves (2.3) in  $S_T$ . Then:*

(i) *For  $\mu \geq \max\{2, mn/2\}$  and  $\frac{1}{p_1} < \frac{1}{p} \leq \frac{1}{2} + \frac{m+6\mu}{2\mu n(m+2)}$ ,*

$$\|w\|_{L_t^s L_x^q(S_T)} \lesssim \|f\|_{L_t^2 L_x^p(S_T)}, \quad (4.8)$$

*where  $q$  and  $s$  are defined in (3.37).*

(ii) *For  $\mu \geq \max\{2, mn/2\}$  and  $\frac{1}{2} \leq \frac{1}{p} < \frac{1}{2} + \frac{2\mu(n-3)+m(3n-1)}{\mu(m+2)(n^2-1)}$ ,*

$$\|w\|_{L_t^2 L_x^q(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)}, \quad (4.9)$$

*where  $r$  is defined in (3.1) and*

$$\frac{1}{q} = \frac{n+1}{2np} + \frac{n-1}{4n} - \frac{m+6\mu}{2\mu(m+2)n}. \quad (4.10)$$

(iii) *For  $\mu \geq \max\{2, \frac{m}{2}\}$  and  $1 < p < p_1$  and  $\gamma = \frac{3}{m+2} - n(\frac{1}{p} - \frac{1}{2})$ ,*

$$\|w(t, \cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)} \lesssim \|f\|_{L_t^2 L_x^p(S_T)}. \quad (4.11)$$

*Proof.* (i) Note that, under these assumptions,

$$1 < \frac{2\mu n(m+2)}{\mu n(m+2) + 6\mu + m} \leq p < p_1, \quad 2 \leq q < \infty, \quad 2 \leq s < \infty.$$

Thus, we get estimate (4.8) by applying Proposition 2.1 and Theorem 3.8.

(ii) This will follow from the dual version of Theorem 3.8. Indeed, when

$$\frac{1}{2} \leq \frac{1}{p} < \frac{1}{2} + \frac{2\mu(n-3) + m(3n-1)}{\mu(m+2)(n^2-1)},$$

then, for  $q$  defined in (4.10),

$$1 < \frac{2\mu(m+2)n}{\mu(m+2)n + 6\mu + m} \leq q' < p_1$$

and

$$\frac{1}{p'} = \frac{2n}{q'(n+1)} - \frac{n-1}{2(n+1)} - \frac{m+6\mu}{\mu(m+2)(n+1)},$$

For  $r$  defined by (3.1), the conjugate exponent  $r'$  can be expressed by

$$r' = \frac{8\mu p'}{\mu(m+2)(n-1)(p'-2) + 2mp'}.$$

Thus, from Remark 3.9, we have that

$$\|W_j^* f\|_{L_t^{r'} L_x^{p'}(S_T)} \lesssim \|f\|_{L_t^2 L_x^{q'}(S_T)},$$

and then, by duality,

$$\|W_j f\|_{L_t^2 L_x^q(S_T)} \lesssim \|f\|_{L_t^r L_x^p(S_T)}.$$

Therefore, from Proposition 2.1 we have that estimate (4.9) holds.

(iii) Note again that  $W_j(|D_x|^\gamma f) = W_j^{-\gamma}(f)$ . Then, in view of (4.6) and estimate (3.20) for  $\alpha = -\gamma = n(\frac{1}{p} - \frac{1}{2}) - \frac{3}{m+2}$ , one has that estimate (4.11) holds.  $\square$

Now we consider the homogeneous equation, i.e., problem (2.2).

**Theorem 4.4.** *Let  $n \geq 2$  and  $\mu \geq \max\{2, m/2\}$ . Suppose  $v$  solves the Cauchy problem (2.2). Then:*

(i) *For  $q_0 \leq q < \infty$ ,*

$$\|v\|_{L^q(\mathbb{R}_+^{1+n})} \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-\frac{2}{m+2}}(\mathbb{R}^n)}, \quad (4.12)$$

*where  $\gamma = \frac{n}{2} - \frac{(m+2)n+2}{q(m+2)}$ .*

(ii) *For  $2 \leq q < \infty$  when  $n = 2$  and  $m = 1$ , and  $2 \leq q < q_1$  when  $n \geq 2$  and  $m \geq 2$  if  $n = 2$ ,*

$$\|v\|_{L_t^s L_x^q(\mathbb{R}_+^{1+n})} \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-\frac{2}{m+2}}(\mathbb{R}^n)}, \quad (4.13)$$

*where*

$$\frac{1}{s} = \frac{(m+2)(n-1)}{4} \left( \frac{1}{2} - \frac{1}{q} \right) + \frac{m}{4\mu}, \quad \gamma = \frac{n+1}{2} \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{m}{2\mu(m+2)}.$$

(iii) *For  $q_1 < q < \infty$  as well as  $n \geq 2$  and  $m \geq 2$  if  $n = 2$ ,*

$$\|v\|_{L_t^2 L_x^q(\mathbb{R}_+^{1+n})} \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-\frac{2}{m+2}}(\mathbb{R}^n)}, \quad (4.14)$$

*where  $\gamma = n(\frac{1}{2} - \frac{1}{q}) - \frac{1}{m+2}$ .*

*Proof.* The goal is to prove that

$$\|v\|_{L_t^\sigma L_x^\rho(\mathbb{R}_+^{1+n})} \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-\frac{2}{m+2}}(\mathbb{R}^n)} \quad (4.15)$$

for certain  $2 \leq \sigma \leq \infty$  and  $2 \leq \rho < \infty$ .

Note that

$$t(1 + \phi(t)|\xi|)^{-\frac{m+4}{2(m+2)}} \leq (1 + \phi(t)|\xi|)^{-\frac{m}{2(m+2)}} |\xi|^{-\frac{2}{m+2}} \leq (1 + \phi(t)|\xi|)^{-\frac{m}{\mu(m+2)}} |\xi|^{-\frac{2}{m+2}}.$$

In order to establish (4.15), from the expression of the function  $v$  in (4.22) together with (2.9) and (2.10) and the estimates of  $b_\ell(t, \xi)$  ( $1 \leq \ell \leq 4$ ) in (2.13) and (2.14), it suffices to show that

$$\|P\varphi\|_{L_t^\sigma L_x^\rho(\mathbb{R}_+^{1+n})} \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)}, \quad (4.16)$$

where the operator  $P$  is of the form

$$(P\varphi)(t, x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)|\xi|)} a(t, \xi) \hat{\varphi}(\xi) d\xi$$

with  $a \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$  and, for any  $(t, \xi) \in \mathbb{R}_+^{1+n}$ ,

$$|\partial_\xi^\beta a(t, \xi)| \lesssim (1 + \phi(t)|\xi|)^{-\frac{m}{\mu(m+2)}} |\xi|^{-|\beta|}. \quad (4.17)$$

Note that  $P\varphi$  can be written as

$$(P\varphi)(t, x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)|\xi|)} a(t, \xi) \widehat{|D_x|^\gamma \varphi}(\xi) \frac{d\xi}{|\xi|^\gamma},$$

and, for  $h = |D_x|^\gamma \varphi$ , by Plancherel's theorem,

$$\|h\|_{L^2(\mathbb{R}^n)} = \| |\xi|^\gamma \hat{\varphi} \|_{L^2(\mathbb{R}^n)} = \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)}.$$

Therefore, in order to prove (4.16), it suffices to show that the operator  $T$ , where

$$(Th)(t, x) = \int_{\mathbb{R}^n} e^{i(x \cdot \xi + \phi(t)|\xi|)} a(t, \xi) \hat{h}(\xi) \frac{d\xi}{|\xi|^\gamma}, \quad (4.18)$$

is bounded from  $L^2(\mathbb{R}^n)$  to  $L_t^\sigma L_x^\rho(\mathbb{R}_+^{1+n})$ . By duality, it suffices to show that the adjoint  $T^*$  of  $T$ ,

$$(T^*f)(x) = \int_0^\infty \int_{\mathbb{R}^n} e^{i(x \cdot \xi - \phi(\tau)|\xi|)} \overline{a(\tau, \xi)} |\xi|^{-\gamma} \hat{f}(\tau, \xi) d\xi d\tau, \quad (4.19)$$

satisfies

$$\|T^*f\|_{L^2(\mathbb{R}^n)} \lesssim \|f\|_{L_t^{\sigma'} L_x^{\rho'}(\mathbb{R}_+^{1+n})}. \quad (4.20)$$

Note that

$$\begin{aligned} \|T^*f\|_{L^2(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n} (T^*f)(x) \overline{(T^*f)(x)} dx \\ &= \int_{\mathbb{R}_+^{1+n}} TT^*f(t, x) \overline{f(t, x)} dt dx \leq \|TT^*f\|_{L_t^\sigma L_x^\rho} \|f\|_{L_t^{\sigma'} L_x^{\rho'}}. \end{aligned}$$

Thus, in order to get (4.20), we only need to show that

$$\|TT^*f\|_{L_t^\sigma L_x^\rho} \lesssim \|f\|_{L_t^{\sigma'} L_x^{\rho'}}. \quad (4.21)$$

From (4.18) and (4.19), we have that

$$TT^*f(t, x) = \int_0^\infty \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau))|\xi|)} a(t, \xi) \overline{a(\tau, \xi)} \hat{f}(\tau, \xi) \frac{d\xi}{|\xi|^{2\gamma}} d\tau.$$

By (4.17), we further have that

$$\left| \partial_\xi^\beta (a(t, \xi) \overline{a(\tau, \xi)}) \right| \lesssim (1 + |\phi(t) - \phi(\tau)| |\xi|)^{-\frac{m}{\mu(m+2)}} |\xi|^{-|\beta|}.$$

Thus, by Proposition 2.1, in order to get (4.21), it suffices to show that

$$\|G_j f\|_{L_t^\sigma L_x^\rho} \lesssim \|f\|_{L_t^{\sigma'} L_x^{\rho'}},$$

where the operator  $G_j$  is defined as

$$G_j f(t, x) = \int_0^\infty \int_{\mathbb{R}^n} e^{i(x \cdot \xi + (\phi(t) - \phi(\tau))|\xi|)} \Theta(|\xi|/2^j) a(t, \xi) \overline{a(\tau, \xi)} \hat{f}(\tau, \xi) \frac{d\xi}{|\xi|^{2\gamma}} d\tau.$$

Note that  $G_j f$  is essentially  $W_j^{2\gamma - \frac{2}{m+2}} f$ . Therefore, in order to get (4.14), it suffices to show that

$$\|W_j^{2\gamma - \frac{2}{m+2}} f\|_{L_t^\sigma L_x^\rho} \lesssim \|f\|_{L_t^{\sigma'} L_x^{\rho'}}. \quad (4.22)$$

We first show (4.12): For  $\gamma = \frac{n}{2} - \frac{n(m+2)+2}{q(m+2)}$  and  $q = q_0$ , we have that

$$\left(2\gamma - \frac{2}{m+2}\right) = \left(\frac{1}{p_0} - \frac{1}{2}\right)(n+1) - \frac{m}{\mu(m+2)} - \frac{2}{m+2}.$$

Thus, we have from estimate (3.3) when  $r = p = p_0$  that

$$\|W_j^{2\gamma - \frac{2}{m+2}}\|_{L^{q_0}(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L^{p_0}(\mathbb{R}_+^{1+n})}. \quad (4.23)$$

On the other hand, from (2.22) and the compact support of  $\Theta$ ,

$$\|W_j^{2\gamma - \frac{2}{m+2}} f\|_{L^\infty(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L^1(\mathbb{R}_+^{1+n})}. \quad (4.24)$$



By interpolation between (4.23) and (4.24), we obtain that

$$\|W_j^{2\gamma - \frac{2}{m+2}} f\|_{L^q(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L^{q'}(\mathbb{R}_+^{1+n})}, \quad q_0 \leq q \leq \infty.$$

where  $q'$  is the conjugate exponent  $q$ . Therefore, we get estimate (4.12).

Next we derive (4.13): Since

$$\frac{1}{s} = \frac{(m+2)(n-1)}{4} \left( \frac{1}{2} - \frac{1}{q} \right) + \frac{m}{4\mu},$$

we can write

$$\frac{1}{s'} = 1 - \frac{(m+2)(n-1)}{4} \left( \frac{1}{q'} - \frac{1}{2} \right) - \frac{m}{4\mu}$$

Thus, when  $\gamma = \frac{n+1}{2} \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{m}{2\mu(m+2)}$ , applying estimate (3.3) for  $\max\{p_1, 1\} < q' \leq 2$ , we have

$$\|W_j^{2\gamma - \frac{2}{m+2}} f\|_{L_t^s L_x^q(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^{s'} L_x^{q'}(\mathbb{R}_+^{1+n})},$$

and, therefore, estimate (4.13) holds.

Finally we prove (4.14): When  $\gamma = n \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{1}{m+2}$ , we have from (3.5) that, for  $p_1 > 1$  and  $1 < q' < p_1$ ,

$$\|W_j^{2\gamma - \frac{2}{m+2}} f\|_{L_t^2 L_x^q(\mathbb{R}_+^{1+n})} \lesssim \|f\|_{L_t^2 L_x^{q'}(\mathbb{R}_+^{1+n})}.$$

Thus, estimate (4.14) holds.  $\square$

Combining Theorems 4.1, 4.3, and 4.4, we obtain the following results:

**Theorem 4.5.** *Let  $u$  solve the Cauchy problem (2.1) in the strip  $S_T$ . Then*

$$\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L_t^s L_x^q(S_T)} \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma - \frac{2}{m+2}}(\mathbb{R}^n)} + \|f\|_{L_t^r L_x^p(S_T)} \quad (4.25)$$

*provided that the exponents  $p, q, r$ , and  $s$  satisfy scaling invariance condition (1.10) and one of the following sets of conditions:*

(i)

$$\begin{cases} \frac{1}{p} - \frac{1}{q} = \frac{4}{(m+2)(n+1)} \left( 1 + \frac{m}{2\mu} \right), \\ \frac{1}{s} = \frac{(m+2)(n-1)}{4} \left( \frac{1}{2} - \frac{1}{q} \right) + \frac{m}{4\mu}, \\ \gamma = \frac{n+1}{2} \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{m}{2\mu(m+2)}, \end{cases}$$

where  $\mu \geq \mu_*$ ,

$$\begin{cases} -\frac{1}{6\mu} < \gamma < \frac{47}{84} + \frac{25}{42\mu} & \text{if } n = 2, m = 1, \\ |\gamma - \gamma_*| < \gamma_d = \frac{2(2\mu - m)(n+1)}{\mu(m+2)(n-1)(2\mu_* - m)} & \text{if } n \geq 3 \text{ or } n = 2, m \geq 2, \end{cases}$$

and

$$\gamma_* = \frac{2}{m+2} + \frac{m}{2\mu(m+2)} - \frac{(2\mu-m)(n+1)}{2\mu(2\mu_*-m)}.$$

(ii)  $n \geq 3$  or  $n = 2, m \geq 2$  and  $r = 2$ ,

$$\begin{cases} \frac{1}{s} = \frac{(m+2)(n-1)}{4} \left( \frac{1}{2} - \frac{1}{q} \right) + \frac{m}{4\mu}, \\ \gamma = \frac{n+1}{2} \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{m}{2\mu(m+2)}, \end{cases}$$

where  $\mu \geq \max\{2, mn/2\}$  and

$$-\frac{m}{2\mu(m+2)} \leq \gamma < \frac{3}{m+2} - \frac{n(2\mu-m)}{\mu(m+2)(n-1)}.$$

(iii)  $n \geq 3$  or  $n = 2, m \geq 2$  and  $s = 2$ ,

$$\begin{cases} \frac{1}{r} = 1 - \frac{m}{4\mu} - \frac{(m+2)(n-1)}{4} \left( \frac{1}{p} - \frac{1}{2} \right), \\ \gamma = n \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{1}{m+2}, \end{cases}$$

where  $\mu \geq \max\{2, mn/2\}$  and

$$\frac{\mu(n+1) - mn}{\mu(m+2)(n-1)} < \gamma < \frac{2}{m+2} + \frac{m}{2\mu(m+2)}.$$

**Remark 4.6.** We can rewrite the conditions of Theorem 4.5 in terms on  $q$ .

(i) For  $\mu \geq \mu_*$ ,

$$\begin{cases} \frac{8}{63} \left( 1 - \frac{4}{\mu} \right) < \frac{1}{q} \leq \frac{1}{2} & \text{if } n = 2, m = 1, \\ \frac{1}{p_2} < \frac{1}{q} + \frac{4}{(m+2)(n+1)} \left( 1 + \frac{m}{2\mu} \right) < \frac{1}{p_1} & \text{if } n \geq 3 \text{ or } n = 2, m \geq 2. \end{cases} \quad (4.26)$$

(ii) For  $\mu \geq \max\{2, mn/2\}$ ,

$$\frac{2n}{(n+1)p_1} - \frac{n-1}{2(n+1)} - \frac{1}{(m+2)(n+1)} \left( 6 + \frac{m}{\mu} \right) < \frac{1}{q} \leq \frac{1}{2}. \quad (4.27)$$

(iii) For  $\mu \geq \max\{2, mn/2\}$ ,

$$\frac{1}{2} - \frac{1}{2(m+2)n} \left( 6 + \frac{m}{\mu} \right) < \frac{1}{q} < \frac{1}{q_1}. \quad (4.28)$$

**Theorem 4.7.** Let  $u$  solve the Cauchy problem (2.1) in the strip  $S_T$ . Then

$$\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L^q(S_T)} \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-\frac{2}{m+2}}(\mathbb{R}^n)} + \||D_x|^{\gamma-\gamma_0} f\|_{L^{p_0}(S_T)} \quad (4.29)$$

provided that the exponents  $p, q, r$ , and  $s$  satisfy (1.10) and  $\mu \geq \max\{2, m/2\}$ ,  $q_0 \leq q < \infty$ , where

$$\gamma = \frac{n}{2} - \frac{n(m+2)+2}{q(m+2)}, \quad \gamma_0 = \frac{2}{m+2} + \frac{m}{2\mu(m+2)} - \frac{n+1}{2} \left( \frac{1}{p_0} - \frac{1}{2} \right).$$

**Corollary 4.8.** Under the conditions of Theorem 4.7, one has

$$\begin{aligned} \|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L^q(S_T)} + \||D_x|^{\gamma - \frac{1}{m+2}} u\|_{L^{q_0^*}(S_T)} \\ \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma - \frac{2}{m+2}}(\mathbb{R}^n)} + \||D_x|^{\gamma - \frac{1}{m+2}} f\|_{L^{p_0^*}(S_T)}, \end{aligned} \quad (4.30)$$

where  $\gamma = \frac{n}{2} - \frac{(m+2)n+2}{q(m+2)}$  and  $q_0^* \leq q < \infty$ .

*Proof.* This follows by combining estimate (4.29) and Remark 4.2 when  $\mu = \mu_*$ .  $\square$

An application of Theorem 4.5 yields:

**Corollary 4.9.** Let  $u$  solve the Cauchy problem

$$\begin{cases} \partial_t^2 u - t^m \Delta u = fg & \text{in } S_T, \\ u(0, \cdot) = \partial_t u(0, \cdot) = 0. \end{cases}$$

Then, for any  $\mu \geq \mu_*$  and  $0 < R \leq \infty$ ,

$$\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T \cap \Lambda_R)} + \|u\|_{L_t^s L_x^q(S_T \cap \Lambda_R)} + \|u\|_{L_t^\infty L_x^\delta(S_T \cap \Lambda_R)} \lesssim \|f\|_{L_t^\sigma L_x^p(S_T \cap \Lambda_R)} \|g\|_{L_t^s L_x^q(S_T \cap \Lambda_R)}, \quad (4.31)$$

where  $q$  is as in (4.26),

$$\rho = \frac{\mu(m+2)(n+1)}{2(2\mu+m)}, \quad \sigma = \frac{\mu(n+1)}{2\mu-mn}, \quad (4.32)$$

$$\frac{1}{s} = \frac{(m+2)(n-1)}{4} \left( \frac{1}{2} - \frac{1}{q} \right) + \frac{m}{4\mu}, \quad \frac{n}{\delta} = \frac{n}{q} + \frac{2}{m+2} \left( \frac{1}{s} - \frac{m}{4\mu} \right), \quad (4.33)$$

and

$$\Lambda_R = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n \mid |x| + \phi(t) < R\}.$$

*Proof.* First we study the case  $R = \infty$ . Note that (4.33) gives that

$$n \left( \frac{1}{2} - \frac{1}{\delta} \right) = \frac{n+1}{2} \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{m}{2\mu(m+2)}.$$

Applying estimate (4.25) in case (i) together with the Sobolev embedding  $\dot{H}^{n(\frac{1}{2}-\frac{1}{\delta})}(\mathbb{R}^n) \hookrightarrow L^\delta(\mathbb{R}^n)$ , we have

$$\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L_t^s L_x^q(S_T)} + \|u\|_{L_t^\infty L_x^\delta(S_T)} \lesssim \|fg\|_{L_t^r L_x^p(S_T)},$$

where  $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ ,  $\frac{1}{r} = \frac{1}{s} + \frac{1}{\sigma}$ . In addition, from Hölder's inequality,

$$\|fg\|_{L_t^r L_x^p(S_T)} \leq \|f\|_{L_t^\sigma L_x^p(S_T)} \|g\|_{L_t^s L_x^q(S_T)}. \quad (4.34)$$

Thus, estimate (4.31) holds for  $R = \infty$ .

Now let  $R < \infty$ . Let  $\chi$  denote the characteristic function of  $S_T \cap \Lambda_R$ . If  $u$  solves  $\partial_t^2 u - t^m \Delta u = fg$  with vanishing initial data and  $u_\chi$  solves  $\partial_t^2 u_\chi - t^m \Delta u_\chi = \chi fg$  with vanishing initial data, then  $u = u_\chi$  in  $S_T \cap \Lambda_R$  due to finite propagation speed (see [22]). Therefore,

$$\begin{aligned} & \|u\|_{C_t^0 \dot{H}_x^\gamma(S_T \cap \Lambda_R)} + \|u\|_{L_t^s L_x^q(S_T \cap \Lambda_R)} + \|u\|_{L_t^\infty L_x^\delta(S_T \cap \Lambda_R)} \\ &= \|u_\chi\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_\chi\|_{L_t^s L_x^q(S_T)} + \|u_\chi\|_{L_t^\infty L_x^\delta(S_T)} \leq \|\chi f\|_{L_t^r L_x^p(S_T)} \|\chi g\|_{L_t^s L_x^q(S_T)}. \end{aligned}$$

Consequently, estimate (4.31) holds.  $\square$

As another application of Theorem 4.5 we have:

**Corollary 4.10.** Let  $u$  be a solution of

$$\begin{cases} \partial_t^2 u - t^m \Delta u = F(v) & \text{in } S_T, \\ u(0, \cdot) = \partial_t u(0, \cdot) = 0. \end{cases}$$

If  $q < \infty$  and  $\frac{1}{m+2} \leq \gamma = \frac{n}{2} - \frac{n(m+2)+2}{q(m+2)} \leq \frac{m+3}{m+2}$ , then

$$\begin{aligned} & \|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L^q(S_T)} + \| |D_x|^{\gamma - \frac{1}{m+2}} u \|_{L^{q_0^*}(S_T)} \\ & \lesssim \|F'(v)\|_{L^{\frac{\mu_*}{2}}(S_T)} \| |D_x|^{\gamma - \frac{1}{m+2}} v \|_{L^{q_0^*}(S_T)}. \end{aligned} \quad (4.35)$$

*Proof.* This follows from estimate (4.30) by taking fractional derivatives. Indeed, for  $0 \leq \gamma - \frac{1}{m+2} \leq 1$ , one has

$$\begin{aligned} & \|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L^q(S_T)} + \| |D_x|^{\gamma - \frac{1}{m+2}} u \|_{L^{q_0^*}(S_T)} \\ & \lesssim \| |D_x|^{\gamma - \frac{1}{m+2}} (F(v)) \|_{L^{p_0^*}(S_T)} \lesssim \|F'(v)\|_{L^{\frac{\mu_*}{2}}(S_T)} \| |D_x|^{\gamma - \frac{1}{m+2}} v \|_{L^{q_0^*}(S_T)}. \end{aligned}$$

$\square$

## 5. SOLVABILITY OF THE SEMILINEAR GENERALIZED TRICOMI EQUATION

In this section, we will apply Theorems 4.5 and 4.7 and Corollaries 4.8 to 4.10 with  $\mu = \mu_*$  to establish the existence and uniqueness of the solution  $u$  of problem (1.1). Thereby, we will use the following iteration scheme: For  $j \in \mathbb{N}_0$ , let  $u_j$  be the solution of

$$\begin{cases} \partial_t^2 u_j - t^m \Delta u_j = F(u_{j-1}) & \text{in } \mathbb{R}_+ \times \mathbb{R}^n, \\ u_j(0, \cdot) = \varphi, \quad \partial_t u_j(0, \cdot) = \psi, \end{cases} \quad (5.1)$$

where  $u_{-1} = 0$ . Notice that, for  $\mu = \mu_*$ , the exponents from (4.25) in case (i) are

$$\gamma_* = \frac{1}{m+2}, \quad \gamma_d = \frac{2(n+1)}{\mu_*(m+2)(n-1)}.$$

In order to get the existence of solutions of the Cauchy problem (1.1) as stated in Theorems 1.1, 1.4, and 1.5, we need to show that, for the sequences  $\{u_j\}_{j=0}^\infty$  and  $\{F(u_j)\}_{j=0}^\infty$  defined by (5.1), there exist a  $T > 0$  and a function  $u$  such that

$$u_j \rightarrow u \quad \text{in } L_{\text{loc}}^1(S_T) \quad \text{as } j \rightarrow \infty, \quad (5.2)$$

$$F(u_j) \rightarrow F(u) \quad \text{in } L_{\text{loc}}^1(S_T) \quad \text{as } j \rightarrow \infty. \quad (5.3)$$

From (5.2) and (5.3), one obviously has that the limit function  $u$  solves problem (1.1) in  $S_T$ .

Furthermore, let  $u, \tilde{u}$  both solve the Cauchy problem (1.1) in  $S_T$ . Then  $v = u - \tilde{u}$  satisfies

$$\begin{cases} \partial_t^2 v - t^m \Delta v = G(u, \tilde{u})v & \text{in } S_T, \\ v(0, \cdot) = \partial_t v(0, \cdot) = 0, \end{cases} \quad (5.4)$$

where  $G(u, \tilde{u}) = \frac{F(u) - F(\tilde{u})}{u - \tilde{u}}$  if  $u \neq \tilde{u}$  and  $G(u, u) = F'(u)$ . For certain  $s, q \geq 2$ , we will show that  $v \in L_t^s L_x^q(S_T)$  and then

$$\|v\|_{L_t^s L_x^q(S_T)} \leq \frac{1}{2} \|v\|_{L_t^s L_x^q(S_T)}. \quad (5.5)$$

Uniqueness of the solution of the Cauchy problem (1.1) in  $S_T$  follows.

### 5.1. Proof of Theorem 1.1.

5.1.1. *Case  $\kappa_1 < \kappa < \kappa_*$ .* From the assumptions of Theorem 1.1, we have

$$\gamma = \frac{n+1}{4} - \frac{n+1}{\mu_*(\kappa-1)} - \frac{m}{2\mu_*(m+2)}$$

and

$$q = \frac{\mu_*(\kappa-1)}{2}, \quad \frac{1}{s} = \frac{(m+2)(n-1)}{4} \left( \frac{1}{2} - \frac{1}{q} \right) + \frac{m}{4\mu_*}. \quad (5.6)$$

Thus,

$$\gamma = \frac{n+1}{2} \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{m}{2\mu_*(m+2)}, \quad \frac{1}{m+2} - \frac{2(n+1)}{\mu_*(m+2)(n-1)} < \gamma < \frac{1}{m+2}.$$

**Existence.** In order to show (5.2), set

$$H_j(T) = \|u_j\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_j\|_{L_t^s L_x^q(S_T)}, \quad N_j(T) = \|u_j - u_{j-1}\|_{L_t^s L_x^q(S_T)}. \quad (5.7)$$

We claim that there exists a constant  $\varepsilon_0 > 0$  small such that

$$2T^{\frac{1}{q} - \frac{1}{s}} H_0(T) \leq \varepsilon_0 \quad (5.8)$$

and

$$H_j(T) \leq 2H_0(T), \quad N_j(T) \leq \frac{1}{2} N_{j-1}(T). \quad (5.9)$$

Indeed, from the iteration scheme (5.1), we have

$$(\partial_t^2 - t^m \Delta)(u_{j+1} - u_{k+1}) = G(u_j, u_k)(u_j - u_k). \quad (5.10)$$

Note that in (4.32)

$$\rho = \sigma = \frac{\mu_*}{2}$$

when  $\mu = \mu_*$ . Thus, from (4.31) and condition (1.2),

$$\begin{aligned} \|u_{j+1} - u_{k+1}\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_{j+1} - u_{k+1}\|_{L_t^s L_x^q(S_T)} \\ \lesssim \|G(u_j, u_k)\|_{L^{\frac{\mu_*}{2}}(S_T)} \|u_j - u_k\|_{L_t^s L_x^q(S_T)} \\ \lesssim (\|u_j\|_{L^q(S_T)}^{\kappa-1} + \|u_k\|_{L^q(S_T)}^{\kappa-1}) \|u_j - u_k\|_{L_t^s L_x^q(S_T)}. \end{aligned} \quad (5.11)$$

Note that  $s > q$  for  $\kappa < \kappa_*$ . By Hölder's inequality, we arrive at

$$\|u_j\|_{L^q(S_T)} \leq T^{\frac{1}{q} - \frac{1}{s}} \|u_j\|_{L_t^s L_x^q(S_T)}. \quad (5.12)$$

Since  $u_{-1} = 0$ , (5.11) together with (5.12) implies that

$$\|u_{j+1} - u_0\|_{L_t^s L_x^q(S_T)} + \|u_{j+1} - u_0\|_{C_t^0 \dot{H}_x^\gamma(S_T)} \lesssim T^{(\kappa-1)(\frac{1}{q}-\frac{1}{s})} \|u_j\|_{L_t^s L_x^q(S_T)}^\kappa.$$

From the Minkowski inequality, we have that there exists an  $\varepsilon_0$  with  $0 < \varepsilon_0 \leq 2^{-2/(\kappa-1)}$  such that

$$H_{j+1}(T) \leq H_0(T) + \frac{1}{2} H_j(T) \quad \text{if } T^{\frac{1}{q}-\frac{1}{s}} H_j(T) \leq \varepsilon_0.$$

Therefore, by induction on  $j$ ,

$$H_j(T) \leq 2H_0(T) \quad \text{if } 2T^{\frac{1}{q}-\frac{1}{s}} H_0(T) \leq \varepsilon_0. \quad (5.13)$$

Taking  $k = j - 1$  in (5.10), estimates (5.11) to (5.13) yield that

$$N_{j+1}(T) \leq \frac{1}{2} N_j(T) \quad \text{if } 2H_0(T)T^{\frac{1}{q}-\frac{1}{s}} \leq \varepsilon_0,$$

which together with (5.13) implies that (5.9) holds as long as (5.8) holds.

Since  $u_{-1} \equiv 0$  and  $u_0$  is a solution of problem (2.2), we have from (4.13) that, for  $\varphi \in \dot{H}^\gamma(\mathbb{R}^n)$  and  $\psi \in \dot{H}^{\gamma-\frac{2}{m+2}}(\mathbb{R}^n)$ ,

$$N_0(T) \leq H_0(T) \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-\frac{2}{m+2}}(\mathbb{R}^n)}.$$

Thus, by choosing  $T > 0$  small, (5.8) holds. Consequently, there is a function  $u \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L_t^s L_x^q(S_T)$  such that

$$u_j \rightarrow u \quad \text{in } L_t^s L_x^q(S_T) \quad \text{as } j \rightarrow \infty, \quad (5.14)$$

and, therefore, (5.2) holds. It also follows that  $u_j$  converges to  $u$  almost where. By Fatou's lemma, it follows that

$$\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L_t^s L_x^q(S_T)} \leq \liminf_{j \rightarrow \infty} \left( \|u_j\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_j\|_{L_t^s L_x^q(S_T)} \right) \leq 2H_0(T), \quad (5.15)$$

which shows that estimate (1.4) holds.

Now we prove (5.3). It suffices to show that  $F(u)$  is bounded in  $L_t^r L_x^p(S_T)$  and  $F(u_j)$  converges to  $F(u)$  in  $L_t^r L_x^p(S_T)$  as  $j \rightarrow \infty$ , where  $p = q/\kappa$  and  $\frac{1}{r} = 1 - \frac{m}{4\mu_*} - \frac{(m+2)(n-1)}{4} \left( \frac{1}{p} - \frac{1}{2} \right)$ . In fact,  $r\kappa < s$  if  $\kappa < \kappa_*$ , thus, for  $q = p\kappa$ , by condition (1.2) and Hölder's inequality, we have

$$\|F(u)\|_{L_t^r L_x^p(S_T)} \lesssim \|u\|_{L_t^{\kappa} L_x^{p\kappa}(S_T)}^\kappa \lesssim T^{\frac{1}{r}-\frac{\kappa}{s}} \|u\|_{L_t^s L_x^q(S_T)}^\kappa.$$

Moreover, in view of  $\frac{1}{p} - \frac{1}{q} = \frac{1}{r} - \frac{1}{s} = \frac{2}{\mu_*}$ , by Hölder's inequality and estimates (5.11)-(5.13) and (5.15), we have

$$\begin{aligned} \|F(u_j) - F(u)\|_{L_t^r L_x^p(S_T)} &\leq \|G(u_j, u)\|_{L^{\mu_*/2}(S_T)} \|u_j - u\|_{L_t^s L_x^q(S_T)} \\ &\lesssim T^{(\kappa-1)(\frac{1}{q}-\frac{1}{s})} H_0(T)^{\kappa-1} \|u_j - u\|_{L_t^s L_x^q(S_T)} \lesssim \|u_j - u\|_{L_t^s L_x^q(S_T)}. \end{aligned}$$

Applying (5.14), we have that  $F(u_j)$  converges to  $F(u)$  in  $L_t^r L_x^p(S_T)$  and, therefore, (5.3) holds.

From (5.2) and (5.3), we have that the limit function  $u \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L_t^s L_x^q(S_T)$  solves the Cauchy problem (1.1) in  $S_T$ .

**Uniqueness.** Suppose  $u, \tilde{u} \in C([0, T], \dot{H}^\gamma(\mathbb{R}^n)) \cap L_t^s L_x^q(S_T)$  solve the Cauchy problem (1.1) in  $S_T$ . Then  $v = u - \tilde{u} \in C([0, T], \dot{H}^\gamma(\mathbb{R}^n)) \cap L_t^s L_x^q(S_T)$  is a solution of problem (5.4). From Corollary 4.9, we have that

$$\begin{aligned} \|v\|_{L_t^s L_x^q(S_T)} &\leq C(\|u\|_{L^q(S_T)}^{\kappa-1} + \|\tilde{u}\|_{L^q(S_T)}^{\kappa-1})\|v\|_{L_t^s L_x^q(S_T)} && \text{(by (4.31) and (1.2))} \\ &\leq CT^{(\kappa-1)(\frac{1}{q}-\frac{1}{s})}(\|u\|_{L_t^s L_x^q(S_T)}^{\kappa-1} + \|\tilde{u}\|_{L_t^s L_x^q(S_T)}^{\kappa-1})\|v\|_{L_t^s L_x^q(S_T)} && \text{(by Hölder's inequality)} \\ &\leq C2^\kappa(T^{\frac{1}{q}-\frac{1}{s}}H_0(T))^{\kappa-1}\|v\|_{L_t^s L_x^q(S_T)} && \text{(by (5.15))} \\ &\leq \frac{1}{2}\|v\|_{L_t^s L_x^q(S_T)}. && \text{(by (5.8))} \end{aligned}$$

Thus (5.5) holds and  $u = \tilde{u}$  in  $S_T$ .

5.1.2. *Case  $\kappa_* \leq \kappa$  if  $n = 2$  or  $\kappa_* \leq \kappa \leq \kappa_3$  if  $n \geq 3$ .*

**Existence.** From the assumptions of Theorem 1.1, we have

$$\gamma = \frac{n}{2} - \frac{4}{(m+2)(\kappa-1)}, \quad s = q = \frac{\mu_*(\kappa-1)}{2}.$$

Thus,

$$\frac{1}{m+2} \leq \gamma = \frac{n}{2} - \frac{(m+2)n+2}{q(m+2)} \leq \frac{m+3}{m+2}.$$

To show (5.2), we set

$$H_j(T) = \|u_j\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_j\|_{L^q(S_T)} + \| |D_x|^{\gamma-\frac{1}{m+2}} u_j \|_{L^{q_0^*}(S_T)},$$

and

$$N_j(T) = \|u_j - u_{j-1}\|_{L^{q_0^*}(S_T \cap \Lambda_R)}. \quad (5.16)$$

We claim that there exists a constant  $\varepsilon_0 > 0$  such that

$$H_0(T) \leq \varepsilon_0, \quad (5.17)$$

and

$$H_j(T) \leq 2H_0(T), \quad N_j(T) \leq \frac{1}{2}N_{j-1}(T). \quad (5.18)$$

Indeed, since  $u_{-1} = 0$ , from the iteration scheme (5.1), we have

$$(\partial_t^2 - t^m \Delta)(u_{j+1} - u_0) = F(u_j). \quad (5.19)$$

Thus, estimate (4.35) together with condition (1.2) yields that, for  $0 \leq \gamma - \frac{1}{m+2} \leq 1$ ,

$$\begin{aligned} H_{j+1}(T) &\leq H_0(T) + C\|F'(u_j)\|_{L^{\frac{\mu_*}{2}}(S_T)} \| |D_x|^{\gamma-\frac{1}{m+2}} u_j \|_{L^{q_0^*}(S_T)} \\ &\leq H_0(T) + C\|u_j\|_{L^q(S_T)}^{\kappa-1} \| |D_x|^{\gamma-\frac{1}{m+2}} u_j \|_{L^{q_0^*}(S_T)} \\ &\leq H_0(T) + CH_j(T)^\kappa. \end{aligned}$$

Therefore, by induction, we have that

$$H_j(T) \leq 2H_0(T) \quad \text{if } C2^\kappa H_0(T)^{\kappa-1} < 1.$$

Consequently,

$$H_j(T) \leq 2H_0(T) \quad \text{if } H_0(T) \leq \varepsilon_0 \quad (5.20)$$

for some  $\varepsilon_0 > 0$  small. Notice that, for  $q$  and  $s$  from (5.6), when  $q = s$ , so  $q = s = q_0^*$ . Hence, by using estimates (5.11)-(5.13) together with (5.20), we get that for  $N_j$  defined in (5.16),

$$N_j(T) \leq \frac{1}{2}N_{j-1}(T) \quad \text{if } H_0(T) \leq \varepsilon_0. \quad (5.21)$$

Estimates (5.20) and (5.21) tell us that (5.18) holds as long as (5.17) holds. To get (5.17), from estimate (4.30) (with  $f = 0$ ) we have that, for  $\varphi \in \dot{H}^\gamma(\mathbb{R}^n)$  and  $\psi \in \dot{H}^{\gamma-\frac{2}{m+2}}(\mathbb{R}^n)$ ,

$$H_0(T) \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-\frac{2}{m+2}}(\mathbb{R}^n)}. \quad (5.22)$$

Due to the continuity of the norm in  $L^q(S_T)$ , (5.17) holds for some  $T > 0$  small. (If  $\|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-\frac{2}{m+2}}(\mathbb{R}^n)}$  is small, then (5.17) holds for any  $T > 0$ , consequently, we get global existence.)

Note that  $q = \mu_*(\kappa - 1)/2 \geq q_0^*$  when  $\kappa \geq \kappa_*$ . Thus, from Hölder's inequality and (5.22),

$$N_0(T) = \|u_0\|_{L^{q_0^*}(S_T \cap \Lambda_R)} \lesssim \|u_0\|_{L^q(S_T)} \lesssim H_0(T). \quad (5.23)$$

From estimates (5.17), (5.18), and (5.23), we get that there exists a function  $u \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L^q(S_T)$  with  $|D_x|^{\gamma-\frac{1}{m+2}}u \in L^{q_0^*}(S_T)$  such that

$$u_j \rightarrow u \quad \text{in } L^{q_0^*}(S_T \cap \Lambda_R) \quad \text{as } j \rightarrow \infty, \quad (5.24)$$

and (5.2) holds. Thus, from Fatou's lemma and (5.18), it follows that

$$\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L^q(S_T)} + \||D_x|^{\gamma-\frac{1}{m+2}}u\|_{L^{q_0^*}(S_T)} \leq 2H_0(T) \quad (5.25)$$

and  $u$  satisfies estimate (1.4).

Since  $q = \mu_*(\kappa - 1)/2 \geq \kappa$  when  $\kappa \geq \kappa_*$ , we have from condition (1.2) that  $F(u)$  is locally integrable for  $u \in L^q(S_T)$ . By Hölder's inequality,

$$\begin{aligned} \int_{S_T \cap \Lambda_R} |F(u_j) - F(u)| dt dx &= \int_{S_T \cap \Lambda_R} |G(u_j, u)| |u_j - u| dt dx \\ &\leq \|G(u_j, u)\|_{L^{p_0^*}(S_T \cap \Lambda_R)} \|u_j - u\|_{L^{q_0^*}(S_T \cap \Lambda_R)}. \end{aligned}$$

Note that  $p_0^* < \mu_*/2$ . Thus, from condition (1.2) we have that

$$\begin{aligned} \|G(u_j, u)\|_{L^{p_0^*}(S_T \cap \Lambda_R)} &\lesssim \|u_j\|_{L^{p_0^*(\kappa-1)}(S_T \cap \Lambda_R)}^{\kappa-1} + \|u\|_{L^{p_0^*(\kappa-1)}(S_T \cap \Lambda_R)}^{\kappa-1} \\ &\lesssim \|u_j\|_{L^q(S_T \cap \Lambda_R)}^{\kappa-1} + \|u\|_{L^q(S_T \cap \Lambda_R)}^{\kappa-1} \lesssim H_0(T)^{\kappa-1}, \end{aligned}$$

which together with (5.24) implies that  $F(u_j) \rightarrow F(u)$  in  $L_{\text{loc}}^1(S_T)$ . Hence, (5.3) holds.

From (5.2) and (5.3), we have that the limit function  $u \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L^q(S_T)$  with  $|D_x|^{\gamma-\frac{1}{m+2}}u \in L^{q_0^*}(S_T)$  is a weak solution of Cauchy problem (1.1) in  $S_T$ .



**Uniqueness.** Suppose  $u, \tilde{u} \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L^q(S_T)$  with  $|D_x|^{\gamma - \frac{1}{m+2}} u, |D_x|^{\gamma - \frac{1}{m+2}} \tilde{u} \in L^{q_0^*}(S_T)$  solve the Cauchy problem (1.1) in  $S_T$ . Then  $v = u - \tilde{u} \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L^q(S_T)$  is a weak solution of problem (5.4). Thus, it follows from Corollary 4.9 that

$$\begin{aligned} \|v\|_{L^q(S_T)} &\leq C \left( \|u\|_{L^q(S_T)}^{\kappa-1} + \|\tilde{u}\|_{L^q(S_T)}^{\kappa-1} \right) \|v\|_{L^q(S_T)} \quad (\text{by (4.31) and (1.2)}) \\ &\leq C 2^\kappa H_0(T)^{\kappa-1} \|v\|_{L^q(S_T)} \quad (\text{by (5.25)}) \\ &\leq \frac{1}{2} \|v\|_{L_t^s L_x^q(S_T)} \quad (\text{by (5.17)}). \end{aligned}$$

Thus (5.5) holds and  $u = \tilde{u}$  in  $S_T$ .

5.1.3. *Case  $n \geq 3$  and  $\kappa > \kappa_3$ ,  $\kappa \in \mathbb{N}$ .*

**Existence.** From the assumptions of Theorem 1.1, we have

$$\gamma = \frac{n}{2} - \frac{4}{(m+2)(\kappa-1)}, \quad s = q = \frac{\mu_*(\kappa-1)}{2}, \quad F(u) = \pm u^\kappa,$$

and

$$\gamma = \frac{n}{2} - \frac{(m+2)n+2}{q(m+2)} > 1 + \frac{1}{m+2}.$$

To verify (5.2), we set

$$H_j(T) = \|u_j\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \sup_{q_0^* \leq \tau \leq \frac{\mu_*(\kappa-1)}{2}} \left\| |D_x|^{\frac{(m+2)n+2}{\tau(m+2)} - \frac{4}{(m+2)(\kappa-1)}} u_j \right\|_{L^\tau(S_T)}$$

and

$$N_j(T) = \|u_j - u_{j-1}\|_{L^{q_0^*}(S_T \cap \Lambda_R)}.$$

We claim that there exists a constant  $\varepsilon_0 > 0$  such that

$$H_0(T) \leq \varepsilon_0 \quad (5.26)$$

and

$$H_j(T) \leq 2H_0(T), \quad N_j(T) \leq \frac{1}{2} N_{j-1}(T). \quad (5.27)$$

In fact, applying Minkowski's inequality and estimate (4.30) (with  $\varphi = \psi = 0$ ),

$$H_{j+1}(T) \leq H_0(T) + C \sup_{q_0^* \leq \tau \leq \mu_*(\kappa-1)/2} \left\| |D_x|^{\frac{n}{2} - \frac{1}{m+2} - \frac{4}{(m+2)(\kappa-1)}} (u_j^\kappa) \right\|_{L^{p_0^*}(S_T)}. \quad (5.28)$$

Note that  $\alpha = \frac{n}{2} - \frac{1}{m+2} - \frac{4}{(m+2)(\kappa-1)} > 1$  when  $\kappa > \kappa_3$ . Thus,  $|D_x|^\alpha (u_j^\kappa)$  can be expressed as a finite linear combination of  $\prod_{\ell=1}^{\kappa} |D_x|^{\alpha_\ell} u_j$ , where  $0 \leq \alpha_\ell \leq \alpha$  ( $1 \leq \ell \leq \kappa$ ) and  $\sum_{\ell=1}^{\kappa} \alpha_\ell = \alpha$ . By Hölder's inequality,  $\left\| |D_x|^\alpha (u_j^\kappa) \right\|_{L^{p_0^*}(S_T)}$  is dominated by a finite sum of terms of the form  $\prod_{\ell=1}^{\kappa} \left\| |D_x|^{\alpha_\ell} u_j \right\|_{L^{\tau_\ell}(S_T)}$ , where  $\sum_{\ell=1}^{\kappa} 1/\tau_\ell = 1/p_0^*$ . We choose  $\tau_\ell$  so that

$$\alpha_\ell = \frac{n(m+2)+2}{\tau_\ell(m+2)} - \frac{4}{(m+2)(\kappa-1)}.$$

Then

$$q_0^* \leq \tau_\ell \leq \frac{\mu_*(\kappa-1)}{2}, \quad \sum_{\ell=1}^{\kappa} \frac{1}{\tau_\ell} = \frac{1}{p_0^*},$$

and, therefore,

$$\| |D_x|^{\alpha_\ell} u_j \|_{L^{\tau_\ell}(S_T)} \leq H_j(T),$$

which together with (5.28) yields that

$$H_{j+1}(T) \leq H_0(T) + C_\kappa H_j(T)^\kappa.$$

By induction, we have that

$$H_j(T) \leq 2H_0(T) \quad \text{if } H_0(T) \leq \varepsilon_0. \quad (5.29)$$

For  $q$  and  $s$  from (5.6), when  $q = s$ , so  $q = s = q_0^*$ . Hence, by estimates (5.11)-(5.13) and together with (5.29), we get that

$$N_j(T) \leq \frac{1}{2} N_{j-1}(T) \quad \text{if } H_0(T) \leq \varepsilon_0. \quad (5.30)$$

From (5.29) and (5.30), we get that (5.27) holds as long as (5.26) holds.

Note that

$$\frac{n(m+2)+2}{\tau(m+2)} - \frac{4}{(m+2)(\kappa-1)} = 0, \quad (5.31)$$

for  $\tau = \mu_*(\kappa-1)/2$  and

$$\frac{n(m+2)+2}{\tau(m+2)} - \frac{4}{(m+2)(\kappa-1)} = \gamma - \frac{1}{m+2}. \quad (5.32)$$

for  $\tau = q_0^*$ . On the other hand, we have from (4.30) (with  $f = 0$ ) that, for  $\varphi \in \dot{H}^\gamma(\mathbb{R}^n)$  and  $\psi \in \dot{H}^{\gamma-\frac{2}{m+2}}(\mathbb{R}^n)$ ,

$$\begin{aligned} \|u_0\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_0\|_{L^{\frac{\mu_*(\kappa-1)}{2}}(S_T)} + \| |D_x|^{\gamma-\frac{1}{m+2}} u_0 \|_{L^{p_0^*}(S_T)} \\ \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-\frac{2}{m+2}}(\mathbb{R}^n)}. \end{aligned} \quad (5.33)$$

By interpolation together with (5.31)-(5.33), we conclude that

$$H_0(T) \lesssim \|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-\frac{2}{m+2}}(\mathbb{R}^n)}.$$

It follows that (5.26) holds by choosing  $T > 0$  small. (We can take  $T = \infty$  if  $\|\varphi\|_{\dot{H}^\gamma(\mathbb{R}^n)} + \|\psi\|_{\dot{H}^{\gamma-\frac{2}{m+2}}(\mathbb{R}^n)}$  is small which then yields global existence.)

From Hölder's inequality and (5.31),

$$N_0(T) = \|u_0\|_{L^{q_0^*}(S_T \cap \Lambda_R)} \leq C_R \|u_0\|_{L^{\frac{\mu_*(\kappa-1)}{2}}(S_T)} \leq C_R H_0(T) < \infty. \quad (5.34)$$

Therefore, we have from (5.27), (5.26), and (5.34) that there exists a function  $u \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L^q(S_T)$  with  $|D_x|^{\gamma-\frac{1}{m+2}} u \in L^{q_0^*}(S_T)$  such that

$$u_j \rightarrow u \quad \text{in } L^{q_0^*}(S_T \cap \Lambda_R) \quad \text{as } j \rightarrow \infty,$$

and, therefore, (5.2) holds. Thus, from Fatou's lemma and (5.27),

$$\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L^q(S_T)} + \| |D_x|^{\gamma-\frac{1}{m+2}} u \|_{L^{q_0^*}(S_T)} \leq 2H_0(T) \quad (5.35)$$

and  $u$  satisfies estimate (1.4).

Note that  $q = \mu_*(\kappa - 1)/2 \geq \kappa$  when  $\kappa > \kappa_3$ . Thus, for  $u \in L^q(S_T)$ , by Hölder's inequality and condition (1.2), we get that  $F(u)$  is locally integrable and  $F(u_j)$  converges to  $F(u)$  in  $L^1_{\text{loc}}(S_T)$ , and hence (5.3) holds.

Applying (5.2), (5.3), it follows that the limit function  $u \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L^q(S_T)$  with  $|D_x|^{\gamma - \frac{1}{m+2}} u \in L^{q_0^*}(S_T)$  is a weak solution of the Cauchy problem (1.1) in  $S_T$ .

**Uniqueness.** This follows from the same arguments as in 5.1.2.  $\square$

**5.2. Proof of Theorem 1.4.** From the assumption of Theorem 1.4, we have

$$\gamma = \frac{n}{2} - \frac{4}{(m+2)(\kappa-1)},$$

$$\frac{1}{q} = \frac{1}{(m+2)(n+1)} \left( \frac{8}{\kappa-1} - \frac{m}{\mu_*} \right) - \frac{n-1}{2(n+1)},$$

and

$$\frac{1}{s} = \frac{(m+2)(n-1)}{4} \left( \frac{1}{2} - \frac{1}{q} \right) + \frac{m}{4\mu_*}.$$

Thus,

$$\gamma = \left( \frac{n+1}{2} \right) \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{m}{2\mu_*(m+2)}$$

and

$$\frac{1}{m+2} \leq \gamma < \frac{1}{m+2} + \frac{2(n+1)}{\mu_*(m+2)(n-1)},$$

where  $\kappa_* \leq \kappa < \kappa_2$ .

To show (5.2), we set

$$H_j(T) = \|u_j\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_j\|_{L_t^s L_x^q(S_T)} + \|u_j - u_0\|_{L_t^\infty L_x^\delta(S_T)}$$

and

$$N_j(T) = \|u_j - u_{j-1}\|_{L_t^s L_x^q(S_T)},$$

where

$$\frac{1}{s} + \frac{(m+2)n}{2q} = \frac{(m+2)n}{2\delta} = \frac{m+2}{2} \left( \frac{n}{2} - \gamma \right). \quad (5.36)$$

We claim that there exist a constant  $\varepsilon_0 > 0$  and a  $\theta \in [0, 1]$  such that

$$2H_0(T)^\theta \left( 2H_0(T) + \|u_0\|_{L_t^\infty L_x^\delta(S_T)} \right)^{1-\theta} \leq \varepsilon_0 \quad (5.37)$$

and

$$H_j(T) \leq 2H_0(T), \quad N_j(T) \leq \frac{1}{2}N_{j-1}(T). \quad (5.38)$$

Indeed, due to (5.36), from Sobolev's embedding theorem we have that

$$\|u(t, \cdot)\|_{L^\delta(\mathbb{R}^n)} \lesssim \|u(t, \cdot)\|_{\dot{H}^\gamma(\mathbb{R}^n)}.$$

Applying Hölder's inequality, we get that

$$\|u_j\|_{L^{\frac{\mu_*(\kappa-1)}{2}}(S_T)} \leq \|u_j\|_{L_t^s L_x^q(S_T)}^\theta \|u_j\|_{L_t^\infty L_x^\delta(S_T)}^{1-\theta},$$

where  $\theta = \frac{2}{n(m+2)+2} + \frac{4n(m+2)}{\mu_*(m+2)(n-1)(q-2)+2mq}$ . Note that  $0 \leq \theta \leq 1$  for  $\gamma \geq \frac{1}{m+2}$ .

By the same arguments as in the proof of Theorem 1.1, we get that (5.37) and (5.38) hold. Consequently, (5.2) and (5.3) also hold. Hence, the limit  $u \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L_t^s L_x^q(S_T)$  of the sequence  $\{u_j\}$  is a solution of the Cauchy problem (1.1) in  $S_T$ . Moreover, by Fatou's lemma and (5.38), we have that

$$\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L_t^s L_x^q(S_T)} \leq 2H_0(T),$$

which together with (5.37) yields that  $u$  satisfies estimate (1.4).

Further, by the same arguments as in the proof of Theorem 1.1, it follows that if both  $u, \tilde{u}$  solve the Cauchy problem (1.1) in  $S_T$ , then  $u = \tilde{u}$  in  $S_T$ .  $\square$

**5.3. Proof of Theorem 1.5.** From the assumptions of Theorem 1.5, we have

$$\gamma = \frac{n+1}{2} \left( \frac{1}{2} - \frac{1}{q} \right) - \frac{m}{2\mu_*(m+2)}$$

and

$$-\frac{m}{2\mu_*(m+2)} \leq \gamma < \frac{1}{m+2} - \frac{2(n+1)}{\mu_*(m+2)(n-1)} = \frac{3}{m+2} - \frac{n(2\mu_* - m)}{\mu_*(m+2)(n-1)}.$$

To verify (5.2), we set

$$H_j(T) = \|u_j\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_j\|_{L_t^s L_x^q(S_T)}, \quad N_j(T) = \|u_j - u_{j-1}\|_{L_t^s L_x^q(S_T)}.$$

Let  $p = q/\kappa$ . Then

$$\frac{2n}{(n+1)p} = \frac{1}{q} + \frac{6\mu + m}{\mu(m+2)(n+1)} - \frac{n-1}{2(n+1)}.$$

Thus we can apply Theorem 4.5 in case (ii) together with Hölder's inequality to find that

$$\begin{aligned} \|u_{j+1} - u_{k+1}\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u_{j+1} - u_{k+1}\|_{L_t^s L_x^q(S_T)} \\ \lesssim \|F(u_j) - F(u_k)\|_{L_t^2 L_x^p(S_T)} \lesssim \|G(u_j, u_k)\|_{L_t^\rho L_x^\sigma(S_T)} \|u_j - u_k\|_{L_t^s L_x^q(S_T)}, \end{aligned}$$

where  $1/\rho = 1/2 - 1/s$ ,  $1/\sigma = 1/p - 1/q = (\kappa - 1)/q$ .

Note that  $s > (\kappa - 1)\rho$  when  $\gamma < \frac{1}{m+2} - \frac{2(n+1)}{\mu_*(m+2)(n-1)}$ . Due to condition (1.2) and Hölder's inequality,

$$\begin{aligned} \|G(u_j, u_k)\|_{L_t^\rho L_x^\sigma(S_T)} &\lesssim \|u_j\|_{L_t^{\rho(\kappa-1)} L_x^q(S_T)}^{\kappa-1} + \|u_k\|_{L_t^{\rho(\kappa-1)} L_x^q(S_T)}^{\kappa-1} \\ &\lesssim T^{\frac{1}{2}-\frac{1}{s}} (\|u_j\|_{L_t^s L_x^q(S_T)}^{\kappa-1} + \|u_k\|_{L_t^s L_x^q(S_T)}^{\kappa-1}). \end{aligned}$$

As in the proof of Theorem 1.1, we get that

$$H_j(T) \leq 2H_0(T), \quad N_j(T) \leq \frac{1}{2}N_{j-1}(T), \quad (5.39)$$

and

$$N_0(T) \leq H_0(T)T^{1/2-\kappa/s} \leq \varepsilon_0, \quad (5.40)$$

for  $\varepsilon_0 > 0$  small by choosing  $T > 0$  small. Therefore, there is a function  $u \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L_t^s L_x^q(S_T)$  such that

$$u_j \rightarrow u \quad \text{in } L_t^s L_x^q(S_T) \quad \text{as } j \rightarrow \infty$$

and (5.2) holds. Combining Fatou's lemma and (5.39), we see that

$$\|u\|_{C_t^0 \dot{H}_x^\gamma(S_T)} + \|u\|_{L_t^s L_x^q(S_T)} \leq 2H_0(T).$$

Together with (5.40) we get that  $u$  satisfies estimate (1.4).

Moreover, since  $2\kappa > s$ , by condition (1.2) and Hölder's inequality, we have that, for  $p = q/\kappa$ ,

$$\|F(u)\|_{L_t^2 L_x^p(S_T)} \lesssim \|u\|_{L_t^{2\kappa} L_x^q(S_T)}^\kappa \lesssim T^{\frac{1}{2} - \frac{\kappa}{s}} \|u\|_{L_t^s L_x^q(S_T)}^\kappa$$

and

$$\begin{aligned} \|F(u_j) - F(u)\|_{L_t^2 L_x^p(S_T)} &\lesssim T^{\frac{1}{2} - \frac{1}{s}} (\|u_j\|_{L_t^s L_x^q(S_T)}^{\kappa-1} + \|u\|_{L_t^s L_x^q(S_T)}^{\kappa-1}) \|u_j - u\|_{L_t^s L_x^q(S_T)} \\ &\lesssim T^{\frac{1}{2} - \frac{1}{s}} H_0(T)^{\kappa-1} \|u_j - u\|_{L_t^s L_x^q(S_T)}, \end{aligned}$$

Therefore,  $F(u) \in L_t^2 L_x^{q/\kappa}(S_T)$  and  $F(u_j) \rightarrow F(u)$  in  $L_t^2 L_x^{q/\kappa}(S_T)$  as  $j \rightarrow \infty$ , hence (5.3) holds. Consequently, the limit function  $u \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L_t^s L_x^q(S_T)$  solves the Cauchy problem (1.1) in  $S_T$ .

Now suppose  $u, \tilde{u} \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L_t^s L_x^q(S_T)$  both solve the Cauchy problem (1.1) in  $S_T$ . Then  $v = u - \tilde{u} \in C_t^0 \dot{H}_x^\gamma(S_T) \cap L_t^s L_x^q(S_T)$  is a solution of Eq. (5.4). Applying Theorem 4.5 in case (ii) and Hölder's inequality, it follows that

$$\begin{aligned} \|v\|_{L_t^s L_x^q(S_T)} &\leq C \|G(u, \tilde{u})v\|_{L_t^2 L_x^p(S_T)} \leq CT^{\frac{1}{2} - \frac{1}{s}} (\|u\|_{L_t^s L_x^q(S_T)}^{\kappa-1} + \|\tilde{u}\|_{L_t^s L_x^q(S_T)}^{\kappa-1}) \|v\|_{L_t^s L_x^q(S_T)} \\ &\leq CT^{\frac{1}{2} - \frac{1}{s}} H_0(T)^{\kappa-1} \|v\|_{L_t^s L_x^q(S_T)} \leq \frac{1}{2} \|v\|_{L_t^s L_x^q(S_T)}. \end{aligned}$$

Thus (5.5) holds and  $u = \tilde{u}$  in  $S_T$ . □

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DEPARTMENT OF MATHEMATICS, NANJING UNIVERSITY, NANJING 210093, CHINA  
*E-mail address:* zhuopingruan@nju.edu.cn

MATHEMATICAL INSTITUTE, UNIVERSITY OF GÖTTINGEN, BUNSENSTR. 3-5, D-37073 GÖTTINGEN, GERMANY  
*E-mail address:* iwitt@uni-math.gwdg.de

SCHOOL OF MATHEMATICAL SCIENCES, NANJING NORMAL UNIVERSITY, NANJING 210023, CHINA  
*E-mail address:* huicheng@nju.edu.cn, 05407@njnu.edu.cn